Problem Set: Hypothesis Testing

Problem 1 (Calibrating a two-sided z-test and computing a p-value)**.** *Suppose an estimator $\hat{\theta}_n$ satisfies*

$$\frac{\hat{\theta}_n - \theta}{\mathrm{se}(\hat{\theta}_n)} \xrightarrow{d} N(0,1).$$

We test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ using

$$T_n = \left| \frac{\hat{\theta}_n - \theta_0}{\mathrm{se}(\hat{\theta}_n)} \right|.$$

(a) Show that choosing the critical value $c_\alpha = z_{1-\alpha/2}$ yields an asymptotic size-α test.

(b) If in a particular application $T_n = 2.31$, report the (asymptotic) two-sided p-value and decide at $\alpha = 0.05$.

Problem 2 (One-sided test: statistic, p-value, and decisions)**.** *Consider $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$ with test statistic*

$$T_n = \frac{\hat{\theta}_n - \theta_0}{\mathrm{se}(\hat{\theta}_n)}.$$

Suppose the observed value is $T_n = 1.72$.

(a) Compute the one-sided p-value for this right-tailed test and state the decision at $\alpha = 0.05$ and $\alpha = 0.01$.

(b) If instead the alternative were $H_1 : \theta < \theta_0$ (left-tailed), what would the one-sided p-value be for the same data?

Problem 3 (Power function of a two-sided z-test with known variance)**.** *Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ with* known σ. *Test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$ using*

$$Z = \frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}}, \qquad \textit{reject if } |Z| > z_{1-\alpha/2}.$$

(a) Derive the power function $\pi(\theta) = P_\theta(\text{reject } H_0)$.

(b) Take $n = 100$, $\sigma = 10$, $\theta_0 = 50$, $\alpha = 0.05$, and evaluate the power at $\theta = 52$.

Problem 4 (Duality: confidence intervals and hypothesis tests)**.** *Suppose an estimator yields $\hat{\theta} = 1.2$ with $\text{se}(\hat{\theta}) = 0.3$.*

(a) Construct a 95% (asymptotic) confidence interval for θ.

(b) Using only the interval from (a), test $H_0 : \theta = 1$ vs. $H_1 : \theta \neq 1$ at $\alpha = 0.05$. Then compute the corresponding two-sided p-value.

(c) Test $H_0 : \theta \geq 1.5$ vs. $H_1 : \theta < 1.5$ at $\alpha = 0.05$ and report the one-sided p-value.

Problem 5 (Design: sample size for desired power (one-sided z-test))**.** *You will test $H_0 : \mu \geq \mu_0$ vs. $H_1 : \mu < \mu_0$ using $\bar{X}$ from $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ with* known σ. *You will reject for small means:*

$$\textit{reject if } \bar{X} < \mu_0 - z_{1-\alpha} \frac{\sigma}{\sqrt{n}} \quad \left(\textit{equivalently } T = -\tfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_{1-\alpha}\right).$$

Derive the smallest n that guarantees power $1 - \beta$ against the point alternative $\mu = \mu_0 - \delta$ ($\delta > 0$).