Problem Set: Estimators

Problem 1 (Bias, Variance, and MSE for Four Mean Estimators)**.** *Let* $X_1, \dots, X_n \overset{iid}{\sim} F$ *with* $E[X] = \mu$ *and* $Var(X) = \sigma^2 < \infty$. *Consider*

$$\hat{\mu}_n^{(1)} = 0, \quad \hat{\mu}_n^{(2)} = X_1, \quad \hat{\mu}_n^{(3)} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \hat{\mu}_n^{(4)} = \frac{1}{n+\lambda}\sum_{i=1}^{n} X_i \quad (\lambda > 0).$$

(a) *Compute* Bias*,* Var*, and* MSE *of each estimator.*

(b) *For fixed* n, λ*, derive a condition on* (μ, σ^2) *under which* $\mathrm{MSE}(\hat{\mu}_n^{(4)}) < \mathrm{MSE}(\hat{\mu}_n^{(3)})$.

(c) *Which of the four estimators are unbiased?*

Problem 2 (Bias of the Plug-in Variance and an Unbiased Fix)**.** *Define the plug-in variance estimator*

$$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \hat{\mu}_n)^2, \qquad \hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(a) *Show that* $E[\hat{\sigma}_n^2] = \left(1 - \frac{1}{n}\right)\sigma^2$*; hence* $\hat{\sigma}_n^2$ *is downward biased with* $\mathrm{Bias} = -\sigma^2/n$.

(b) *Deduce an unbiased estimator for* σ^2 *and name it.*

Problem 3 (Consistency via WLLN and CMT)**.** *With the four mean estimators from Problem 1 and* $\hat{\sigma}_n^2$ *from Problem 2:*

(a) *Determine which of* $\hat{\mu}_n^{(1)}, \hat{\mu}_n^{(2)}, \hat{\mu}_n^{(3)}, \hat{\mu}_n^{(4)}$ *are consistent for* μ.

(b) *Show* $\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$.

(c) *Suppose* $\lambda = \lambda_n$ *may depend on* n*. Give a sufficient condition on* (λ_n) *for* $\hat{\mu}_n^{(4)}$ *to be consistent.*

Problem 4 (CLT, Delta Method, Slutsky, and Confidence Intervals)**.** *Assume* $X_1, \dots, X_n \overset{iid}{\sim} F$ *with* $E[X] = \mu$ *and* $0 < \sigma^2 = Var(X) < \infty$.

(a) *State the CLT for* $\hat{\mu}_n$ *and give the asymptotic distribution of* $\sqrt{n}(\hat{\mu}_n - \mu)$.

(b) *Use Slutsky's theorem to show*

$$\sqrt{n}\,\frac{\hat{\mu}_n - \mu}{\hat{\sigma}_n} \xrightarrow{d} N(0,1),$$

where $\hat{\sigma}_n$ *is any consistent estimator of* σ *(e.g.* s_n*).*

(c) *Let* $g(m) = m^2$*. Use the Delta Method to obtain the asymptotic distribution of* $\hat{\theta}_n \equiv g(\hat{\mu}_n) = \hat{\mu}_n^2$ *about* $\theta = g(\mu) = \mu^2$.

(d) *Using (b), write a large-sample two-sided* $(1-\alpha)$ *confidence interval for* μ*. Using (c), give a first-order* $(1-\alpha)$ *CI for* μ^2 *(plug-in all unknowns).*