Problem Set: Expectation

Problem 1 (Continuous LOTUS and Quantiles)**.** *Let* $X \sim U(a, b)$ *with* $a < b$ *and define* $Z = \mathbb{1}\{X \leq t\}$ *for some* $t \in \mathbb{R}$*.*

(a) Compute $E[X]$ *and* $Var(X)$*.*

(b) Using LOTUS, compute $E[Z]$ *and interpret it. For which values of* t *does your formula change?*

(c) Let $q \in (0, 1)$*. Show that the* q*-quantile of* X *is* $F_X^{-1}(q) = a + (b - a)q$*. Use this to compute the median.*

Problem 2 (Covariance, Correlation, and Variance of a Sum)**.** *Consider the bivariate discrete random vector* (X, Y) *with joint pmf*

	$Y = 0$	$Y = 1$
$X = 0$	$\frac{1}{5}$	$\frac{1}{10}$
$X = 1$	$\frac{3}{10}$	$\frac{2}{5}$

(a) Compute the marginals $P(X = x)$ *and* $P(Y = y)$*; then compute* $E[X]$*,* $E[Y]$*,* $Var(X)$*, and* $Var(Y)$*.*

(b) Compute $Cov(X, Y)$ *and* $\mathrm{corr}(X, Y)$*.*

(c) Compute $Var(X + Y)$ *directly from the joint pmf, and verify* $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$*.*

(d) Are X *and* Y *independent? Justify concisely.*

Problem 3 (Conditional Expectation/Variance, LIE, and LTV). *Let* $X \sim U(0,1)$ *and, conditional on* X, *let* $Y \mid X \sim \text{Bernoulli}(X)$ *(i.e.,* $P(Y = 1 \mid X) = X$*).*

(a) Compute $E[Y \mid X]$ *and* $Var(Y \mid X)$.

(b) Use the Law of Iterated Expectations (LIE) to compute $E[Y]$.

(c) Use the Law of Total Variance (LTV) to compute $Var(Y)$ *by evaluating* $E[Var(Y \mid X)]$ *and* $Var(E[Y \mid X])$ *separately.*

(d) Compute $Cov(X, Y)$ *and* $\text{corr}(X, Y)$.

Problem 4 (Mean Independence vs. Independence). *Construct random variables with* mean independence without independence. *Let* $X \sim \text{Bernoulli}(1/2)$ *taking values* $\{0, 1\}$. *Define*

$$Y \mid X = \begin{cases} \textit{Uniform}(-2, 2), & \textit{if } X = 0, \\ \textit{Takes values } \{-3, +3\} \textit{ with prob. } 1/2 \textit{ each}, & \textit{if } X = 1. \end{cases}$$

(a) Show that $E[Y \mid X] = 0$ *and compute* $E[Y]$.

(b) Prove Y *is* not *independent of* X *by finding an event* A *such that* $P(Y \in A \mid X) \neq P(Y \in A)$.

(c) Conclude that Y *is mean independent of* X *but not independent of* X.