Problem Set: Probability

Problem 1 (LOTUS)**.** *Let X be the number of heads in two tosses of a fair coin.*

(a) *Find the support and pmf $f_X(x) = P(X = x)$.*

(b) *Compute the CDF $F_X(x) = P(X \leq x)$ for all real x.*

(c) *Compute the median of X, i.e., any m with $F_X(m) \geq 1/2$ and $P(X < m) \leq 1/2$. (You may also use the quantile definition $F_X^{-1}(q) = \inf\{x : F_X(x) \geq q\}$.)*

(d) *Using the* law of the unconscious statistician *(LOTUS), compute $E[g(X)]$ for $g(x) = x^2$.*

Problem 2 (Continuous Uniform, CDF, PDF, and Quantiles)**.** *Let $X \sim U(a, b)$ with $a < b$.*

(a) *Write the pdf f_X, the CDF F_X, and the quantile function F_X^{-1}.*

(b) *Compute $P\big(a < X \leq \frac{a+b}{2}\big)$.*

(c) *Give a concrete example (choose a, b) where the pdf takes values greater than 1. Explain why this does not contradict $P(X = x) = 0$ for continuous X.*

Problem 3 (Joint pmf, Marginals, Conditionals, and Independence)**.** *Consider the discrete bivariate random vector (X, Y) with joint pmf*

	$Y = 0$	$Y = 1$
$X = 0$	$\frac{1}{5}$	$\frac{1}{10}$
$X = 1$	$\frac{3}{10}$	$\frac{2}{5}$

(a) *Compute the marginals $f_X(x)$ and $f_Y(y)$.*

(b) *Compute $P(X = 0 \mid Y = 0)$ and $P(Y = 1 \mid X = 1)$.*

(c) *Check independence by comparing $f_{X,Y}(x, y)$ to $f_X(x)f_Y(y)$ for all support points. Conclude whether X and Y are independent.*

Problem 4 (Bivariate Normal: Standardization and Conditioning)**.** *Suppose* (X, Y) *is bivariate normal with means* (μ_X, μ_Y)*, variances* (σ_X^2, σ_Y^2)*, and covariance* σ_{XY}*.*

(a) Show that $Z_X = (X - \mu_X)/\sigma_X \sim N(0, 1)$ *and express* $P(a < X \le b)$ *in terms of the standard normal CDF* Φ*.*

(b) Compute $P\big(Y \le \mu_Y \,\big|\, X = \mu_X\big)$*, given the distribution of* $Y \mid X = x$*,*

$$Y \mid X = x \sim N\left(\mu_Y + \frac{\sigma_{XY}}{\sigma_X^2}(x - \mu_X), \quad \sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2}\right).$$

(c) State a necessary and sufficient condition for X *and* Y *to be independent in the bivariate normal case, and justify briefly.*