BUSS386 Problem Set 13 — Solutions

Exotic Options

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Problem 1 — Binary as a building block

$S_0 = 100$, $K = 100$, $r = 0.04$, $\sigma = 0.25$, $T = 0.5$. $\sigma\sqrt{T} = 0.17678$.

(a) $d_1 = \dfrac{0 + (0.04 + 0.03125)(0.5)}{0.17678} = \dfrac{0.035625}{0.17678} = \boxed{0.2015}$. $d_2 = 0.2015 - 0.1768 = \boxed{0.0247}$.

(b) $N(d_1) = 0.5799$, $N(d_2) = 0.5099$, $e^{-rT} = 0.9802$.

- Cash-or-nothing call: $e^{-rT}N(d_2) = 0.9802 \times 0.5099 = \boxed{\$0.4998}$ per \$1 of digital payoff.
- Asset-or-nothing call: $S_0\,N(d_1) = 100 \times 0.5799 = \boxed{\$57.99}$.

(c) Vanilla call = AoN call $-K\times$ CoN call $= 57.99 - 100 \times 0.4998 = 57.99 - 49.98 = \boxed{\$8.01}$.

(d) Gap calls pay $S_T - K_1$ when $S_T > K_2$ — exactly 1 asset-or-nothing call at trigger $K_2 - K_1$ cash-or-nothing calls at trigger K_2. Same decomposition, different mix.

Problem 2 — Gap call vs vanilla call

(a) Gap call pays 0 for $S_T \le 110$ then jumps to $S_T - 100$ for $S_T > 110$ (a vertical jump of \$10 at $S_T = 110$). Vanilla call pays $S_T - 100$ for all $S_T > 100$. They differ on $S_T \in (100, 110]$, where vanilla pays positive and gap pays 0.

(b) For the gap, d_1, d_2 use $K_2 = 110$:

$$d_1 = \frac{\ln(100/110) + (0.04 + 0.03125)(0.5)}{0.17678} = \frac{-0.0953 + 0.0356}{0.17678} = \frac{-0.0597}{0.17678} = -0.3376.$$

$d_2 = -0.3376 - 0.1768 = -0.5144$. $N(d_1) = 0.3678$, $N(d_2) = 0.3035$.

$$C_{\text{gap}} = 100 \times 0.3678 - 100 \times 0.9802 \times 0.3035 = 36.78 - 29.75 = \boxed{\$7.03}.$$

(c) Vanilla call at $K = 100$: from Problem 1, $\boxed{\$8.01}$.

(d) Vanilla > gap: vanilla pays an extra $S_T - 100 \in (0, 10]$ on $S_T \in (100, 110]$, whereas the gap pays 0 in that region. Premium difference $= 8.01 - 7.03 =$ \$0.98 is the discounted RN expectation of the missing payoff.

Problem 3 — Compound call (binomial tree)

$u = 1.2$, $d = 0.8$, $\Delta t = 0.5$, $r = 5\%$.

(a) $p = \dfrac{e^{0.025} - 0.8}{1.2 - 0.8} = \dfrac{1.02532 - 0.8}{0.4} = \boxed{0.5633}$. $e^{-r\Delta t} = 0.9753$.

(b) Terminal stock prices: $S_{uu} = 144$, $S_{ud} = 96$, $S_{dd} = 64$. Inner-call payoffs at T: $C_{uu} = 44$, $C_{ud} = 0$, $C_{dd} = 0$.

- $C_u = e^{-r\Delta t}\big(p \cdot 44 + (1-p)\cdot 0\big) = 0.9753 \cdot 0.5633 \cdot 44 = \boxed{24.17}$.
- $C_d = e^{-r\Delta t}\big(p \cdot 0 + (1-p)\cdot 0\big) = \boxed{0}$.

(c) $\text{CoC}_u = \max(24.17 - 10, 0) = \boxed{14.17}$. $\text{CoC}_d = \max(0 - 10, 0) = \boxed{0}$.

(d) $\text{CoC}_0 = e^{-r\Delta t}\big(p \cdot 14.17 + (1-p)\cdot 0\big) = 0.9753 \cdot 0.5633 \cdot 14.17 = \boxed{7.78}$.

Problem 4 — Barrier parity

(a) Parity: $p_{\text{vanilla}} = p_{DI} + p_{DO}$, i.e., a vanilla put equals the sum of its knock-in and knock-out components at the same barrier.

(b) $p_{DO} = p_{\text{vanilla}} - p_{DI} = 8.50 - 4.20 = \boxed{₩\,4.30}$ per index point.

(c) As $H \to K$, more paths breach the barrier $\Rightarrow$ down-and-in price *rises* and approaches vanilla; equivalently, $p_{DO} \to 0$.

(d) A knock-out put dies if the barrier is touched, removing some of the most-ITM payoff scenarios; fewer expected payoffs $\Rightarrow$ lower price than vanilla.

Problem 5 — Korean autocallable ELS (conceptual)

(a) The issuer's embedded position:

- $\boxed{\text{Long a zero-coupon bond}}$ (the principal repayment),
- $\boxed{\text{Short a down-and-in put}}$ at the 50% KI barrier (the principal-at-risk feature),
- $\boxed{\text{Short a strip of Bermudan autocall options}}$ (the early-redemption right at each observation date).

The investor holds the mirror of this.

(b) Short put $\Rightarrow$ $\boxed{\text{short gamma, short vega}}$. The issuer is also $\boxed{\text{short the smile}}$: when OTM-put IV *rises*, the value of the short put *grows*, so the issuer's MTM *falls*.

(c) Q1 2024: the volatility smile and skew for HSCEI options *re-priced* (steeper put skew, higher OTM-put IV) before HSCEI itself moved further. Since the issuer is short the smile, the MTM mark-to-model loss arrived first; realized losses on the underlying came later as the index fell through the 50% KI level.

(d) Multiple Bermudan early-exercise opportunities + path-dependent KI barrier monitored over the life + (usually) worst-of basket $\Rightarrow$ no closed form. Monte Carlo handles all three naturally.

Problem 6 — Asian options

(a) Averaging dampens the variance of the payoff (the average has lower volatility than any single S_t), and lower vol $\Rightarrow$ lower call value.

(b) For any sample path, geometric average $\leq$ arithmetic average (AM–GM inequality); so geometric Asian call payoff $\leq$ arithmetic Asian call payoff path-by-path; hence geometric is cheaper.

(c) (i) A single 1-year European put only protects against the year-end exchange rate, not against weak EUR throughout the year — the firm receives €100m *every month*, exposed to twelve different monthly FX prints. The Asian put averages across all twelve, matching the exposure.

(ii) A basket of 12 monthly European puts is a valid hedge but more expensive than the Asian (basket of 12 individually averaged is less efficient than one option on the average), because each individual option includes its own time value and volatility premium.

Problem 7 — Monte Carlo convergence and KIKO

(a) 95% CI $\approx \hat{V}_0 \pm 2\,\text{s.e.} = 8.40 \pm 2(0.42) = \boxed{[7.56,\ 9.24]}$.

(b) s.e. $\propto 1/\sqrt{N}$. To shrink from 0.42 to 0.05, ratio $= 0.42/0.05 = 8.4$, so $N^* = 1000 \times 8.4^2 = 1000 \times 70.56 = \boxed{\approx 70{,}560 \text{ paths}}$.

(c) The leveraged knock-in leg of KIKO has very large payoff conditional on the barrier being breached. In a naive simulation under the realized-volatility regime of 2007, very few paths breach $\Rightarrow$ the rare-but-large loss path is undersampled $\Rightarrow$ MC underestimates the option's true risk. Importance sampling (oversample tail paths) is essential.

(d) Use the $\boxed{\text{geometric Asian as a control variate}}$: the geometric Asian has a closed-form price, is highly correlated with the arithmetic Asian, and dramatically reduces the variance of the estimator without bias.