BUSS386 Problem Set 13

Exotic Options

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Problem 1 — Binary as a building block

A non-dividend stock has $S_0 = 100$, $r = 4\%$ c.c., $\sigma = 25\%$, $T = 0.5$. Consider digitals at strike $K = 100$.

(a) Compute d_1 and d_2.

(b) Compute the prices of the cash-or-nothing call (pays \$1 if ITM) and the asset-or-nothing call (pays S_T if ITM).

(c) Show that the portfolio (long 1 asset-or-nothing call − short $K = 100$ cash-or-nothing calls) reproduces the BSM vanilla call price. State the price.

(d) In one sentence, why is this decomposition useful for pricing gap options?

Problem 2 — Gap call vs vanilla call

Stock $S_0 = 100$, $r = 4\%$ c.c., $\sigma = 25\%$, $T = 0.5$, $q = 0$. Consider a gap call with trigger $K_2 = 110$ and payout strike $K_1 = 100$.

(a) Sketch the gap-call payoff as a function of S_T. Identify the region where it differs from a vanilla call at $K = K_1 = 100$.

(b) Compute the gap call price (use the formula with d_1, d_2 evaluated at $K_2 = 110$).

(c) Compute the vanilla call price at $K = 100$.

(d) State which is more expensive and explain in one sentence using the payoff region from (a).

Problem 3 — Compound call (binomial tree)

Two-step CRR tree: $S_0 = 100$, $u = 1.2$, $d = 0.8$, $\Delta t = 0.5$, $r = 5\%$ c.c. A vanilla 1-year European call has strike $K_{\text{inner}} = 100$. A compound *call-on-call* expires at $t = 0.5$ with outer strike $K_{\text{outer}} = 10$.

(a) Compute the risk-neutral probability p and per-step discount $e^{-r\Delta t}$.

(b) Compute the inner-call payoffs at $t = T$ and back out C_u, C_d at $t = 0.5$.

(c) Compute the compound-call payoffs $\text{CoC}_u = \max(C_u - K_{\text{outer}}, 0)$ and CoC_d at $t = 0.5$.

(d) Back out CoC_0 at $t = 0$.

Problem 4 — Barrier parity

A 6-month KOSPI 200 European put with $K = 360$ trades at ₩ 8.50 per index point. The corresponding down-and-in put with barrier $H = 324$ trades at ₩ 4.20.

(a) State the barrier parity relation between vanilla, down-and-in, and down-and-out puts.

(b) Compute the price of the down-and-out put.

(c) As H moves from ₩ 324 closer to the strike 360, what happens to the down-and-in price? In one sentence, why?

(d) In one sentence, why does a knock-out option cost less than a vanilla?

Problem 5 — Korean autocallable ELS (conceptual)

A 3-year KOSPI 200–linked autocallable ELS pays an 8% annual coupon. It is autocalled at par if KOSPI 200 closes above 95% of issue level on any of the six semi-annual observation dates. At maturity, if KOSPI 200 has never touched 50% of issue level, the principal is returned; otherwise, the investor takes the index loss (1:1).

(a) From the issuer (bank)'s perspective, decompose the embedded position into building blocks: long/short which standard and exotic instruments?

(b) Is the issuer long or short gamma? Long or short vega? Long or short the smile (i.e., does the issuer benefit when OTM-put IV rises)?

(c) During the HSCEI ELS crisis of Q1 2024, mark-to-market losses preceded realized losses. In one sentence, why?

(d) In one sentence, why is this product priced with Monte Carlo and not closed-form BSM?

Problem 6 — Asian options

KOSPI 200 spot $S_0 = 360$, $r = 3\%$ c.c., $q = 1.5\%$ continuous dividend, $T = 1$ year, $\sigma = 20\%$.

(a) In one sentence, explain why an arithmetic-average Asian call with strike $K = 360$ is cheaper than the corresponding European call.

(b) In one sentence, explain why the geometric Asian call is cheaper still.

(c) A Korean exporter receives € 100m monthly from German customers. State, in one sentence each, why an arithmetic-average Asian put on EUR/KRW is a better hedge than (i) a single 1-year European put on EUR/KRW; (ii) a basket of 12 monthly European puts.

Problem 7 — Monte Carlo convergence and KIKO

(a) A Monte Carlo estimator with $N = 1{,}000$ paths gives $\hat{V}_0 = 8.40$ and s.e. $= 0.42$. State the 95% confidence interval.

(b) How many paths N^* are required to shrink the s.e. to 0.05?

(c) Explain in one sentence why KIKO options are pathological for naive Monte Carlo (think: which path region contributes most to the option's loss and how is it sampled?).

(d) In one sentence, name one variance-reduction technique you would use for an arithmetic-average Asian option and why.