

# Exotic Options

BUSS386. Futures and Options

Prof. Ji-Woong Chung

# Lecture Outline

- What are exotic options?
- Valuation framework (4-way taxonomy)
- Path-independent: binary, compound, gap
- Path-dependent (path-independent valuation): barrier, lookback, Bermudan
- Path-dependent (path-dependent valuation): Asian options
- Multivariate: exchange, quanto
- Pricing exotics: trees + Monte Carlo

# Exotic Options

# Exotic Options

- **Definition.** Nonstandard option contracts whose payoffs differ from the simple call/put structure — typically engineered by “tweaking” standard options.
- **Solve specific business problems** that European/American calls and puts cannot:
  - Hedge only if a price barrier is breached (KIKO).
  - Settle on an *average* price, not a snapshot.
  - Cross-asset payoffs (relative performance, baskets, FX-quanto).
- **Who sells them?** Investment banks and structured-products desks. The bank charges a fee for structuring and dynamic hedging; the client gets a tailored risk profile.

# Four Questions to Ask About Any Exotic

- The goal is *not* to memorize formulas for each exotic option.
  - Instead, focus on intuition: how the payoff works and why the exotic structure is needed.
- **Key questions to ask about any exotic option:**
  - ① **What problem does this exotic option solve?** (Hedging, cost reduction, targeted payoff, path dependence, barrier protection, etc.)
  - ② **Can it be approximated or replicated using a portfolio of standard options?**
    - If yes, this often provides intuition and may help adapt/extend the BSM framework.
  - ③ **Is it cheap or expensive relative to a standard option that provides a similar payoff?** (Important for traders evaluating relative value.)
  - ④ **How do we price it when no closed-form formula exists?**
    - Simulation (Monte Carlo)
    - Binomial/trinomial trees
    - PDE methods
    - Replicating portfolios

# Path-Independent Exotics

Binary, Compound, Gap

# Binary (Digital) Options

## Path-Independent Exotic Options

- **Binary options** pay a fixed amount if the underlying ends in the money — otherwise zero. Pure directional bets, or building blocks for structured payoffs.
- **Cash-or-Nothing.** Pay \$1 if ITM, else 0.

$$C_{\text{con}} = e^{-r(T-t)} N(d_2), \quad P_{\text{con}} = e^{-r(T-t)} N(-d_2)$$

- **Asset-or-Nothing.** Pay  $S_T$  if ITM, else 0.

$$C_{\text{aon}} = S e^{-q(T-t)} N(d_1), \quad P_{\text{aon}} = S e^{-q(T-t)} N(-d_1)$$

- **Quick check.** For a vanilla European call,

$$\underbrace{S e^{-qT} N(d_1)}_{\text{asset-or-nothing call}} - \underbrace{K e^{-rT} N(d_2)}_{K \text{ cash-or-nothing calls}} = \text{vanilla call.}$$

**Every BSM call is a portfolio of two digitals.**

# Binary Options as Building Blocks

## Path-Independent Exotic Options

- Consider the portfolio:
  - Long 1 asset-or-nothing call at strike  $K$ ,
  - Short  $K$  cash-or-nothing calls at strike  $K$ .
- **Payoff at maturity:**
  - If  $S_T > K$ : payoff =  $S_T - K \cdot 1 = S_T - K$ .
  - If  $S_T \leq K$ : payoff =  $0 - K \cdot 0 = 0$ .
- This is exactly the **standard European call** payoff:

$$\max(S_T - K, 0).$$

- **Why this matters.** Many exotic structures (gap options, range accruals, autocallable coupons) decompose into a portfolio of binaries. Pricing them reduces to pricing digitals.

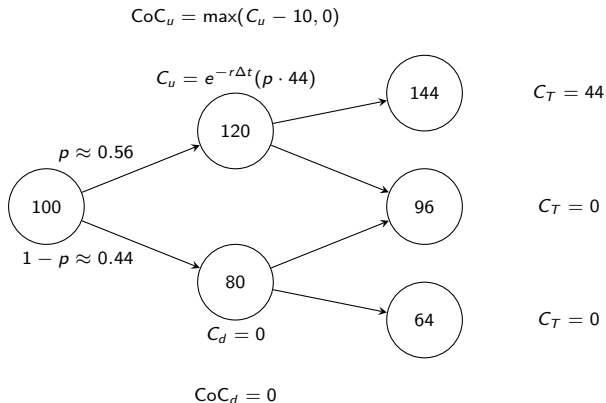
# Compound Options

## Path-Independent Exotic Options

- **Definition.** An option whose underlying is *another option*.
- **Four types:**
  - Call-on-Call (CoC), Put-on-Call (PoC), Call-on-Put (CoP), Put-on-Put (PoP).
- **When useful.** Buyer is uncertain whether they will need the inner option, and wants to defer the premium decision.
- **Corporate use case.** A firm bidding on a 2-year, \$200M infrastructure project:
  - Risk: rates may rise between today and the contract award date.
  - Buying a 2-year **interest rate cap** today hedges this risk — but \$\$\$ and useless if the firm loses the bid.
  - Solution: **call on the cap (CoC)**.
    - Win the bid  $\Rightarrow$  exercise the call, buy the cap, lock the rate.
    - Lose the bid  $\Rightarrow$  compound option expires worthless, lose only the small premium.
- **Pricing.** Closed form exists (Geske 1979) for CoC; in practice, binomial trees with backward induction.

# Binomial Tree for the Compound Call

- $S = K = 100$ ,  $u = 1.2$ ,  $d = 0.8$ ,  $\Delta t = 0.5$ ,  $r = 5\%$
- Risk-neutral probability:  $p = \frac{e^{r\Delta t} - d}{u - d} \approx \frac{1.0253 - 0.8}{0.4} \approx 0.5633$ ,  $1 - p \approx 0.4367$
- Discount factor per step:  $e^{-r\Delta t} \approx 0.9753$



$$\text{Co}C_0 = e^{-r\Delta t}(p \text{Co}C_u + (1 - p) \text{Co}C_d) \Rightarrow \text{Co}C_0 \approx 7.8$$

# Gap Options

## Path-Independent Exotic Options

- A **gap call** has *two* strikes:
  - **Trigger**  $K_2$ : determines *whether* it pays.
  - **Payout strike**  $K_1$ : determines *how much* it pays.
- **Payoff at maturity:**

$$\text{Gap Call} = \begin{cases} S_T - K_1, & S_T > K_2, \\ 0, & S_T \leq K_2. \end{cases}$$

- **Valuation:**

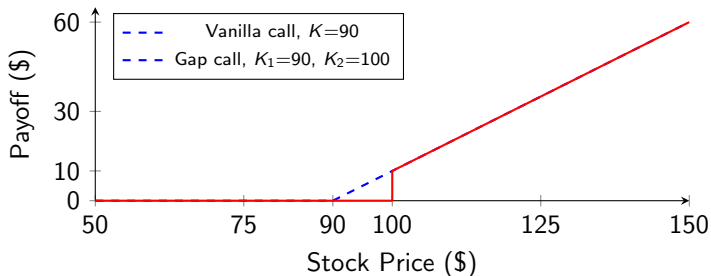
$$C_{\text{gap}} = Se^{-qT} N(d_1) - K_1 e^{-rT} N(d_2), \quad d_1 = \frac{\ln(S/K_2) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

Note: the *trigger*  $K_2$  enters  $d_1, d_2$ ; the *payout*  $K_1$  enters the price.

- **Where used.** Employee stock comp (vesting threshold), structured products where activation and payout are separable.

## Gap Call vs Vanilla Call

- Pays  $S_T - K_1$  when  $S_T > K_2$ . Take  $K_1 = 90$ ,  $K_2 = 100$ .
- Compare against a vanilla call with  $K = K_1 = 90$ .



**Answer.** Gap call < vanilla call: between  $S_T = 90$  and 100, vanilla pays  $S_T - 90 > 0$  while gap pays 0.

# Path-Dependent Exotics

(Valuation is still path-independent)

# Barrier Options

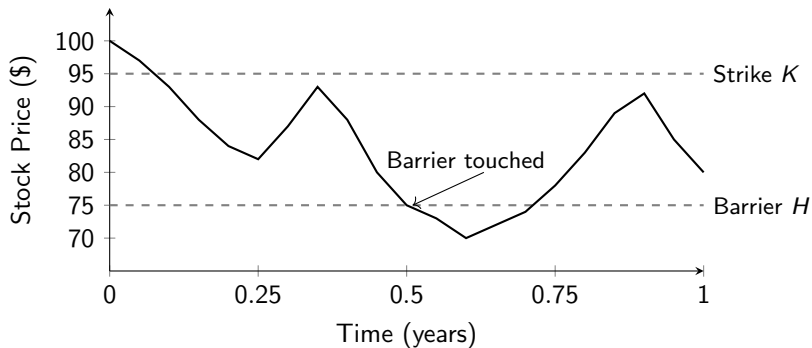
- **Barrier options** activate or extinguish depending on whether the underlying crosses a *barrier* during the option's life.
- **Knock-Out** — cease to exist if the barrier is hit.
  - Down-and-out: dies if  $S$  falls below  $H$ .
  - Up-and-out: dies if  $S$  rises above  $H$ .
- **Knock-In** — come into existence only if the barrier is touched.
  - Down-and-in: activates if  $S$  falls below  $H$ .
  - Up-and-in: activates if  $S$  rises above  $H$ .
- **Rebate** — fixed cash payment if the barrier is hit.
- **Parity:** knock-in + knock-out = vanilla.

$$c = c_{DI} + c_{DO} = c_{UI} + c_{UO}, \quad p = p_{DI} + p_{DO} = p_{UI} + p_{UO}.$$

Compute one component; back out the other.

- **Why used?** Cheaper than vanilla (extra condition reduces payoff probability). Common in structured retail products, FX hedges (incl. KIKO), and exotic equity baskets.

## Down-and-In Barrier Path



Down-and-in put: activated once the path touches  $H$ . Finishes ITM since  $S_T = 80 < K = 95$ .

# Barrier Option Pricing

- **Methods.**

- Modified BSM (closed form for European barriers, continuous monitoring).
- Binomial / trinomial trees: must track whether barrier has been touched.
- Monte Carlo: natural for knock-out; for knock-in, simulate paths and check first-passage.

- **Example: European put on EUR/USD.**

$x_0 = 0.90$ ,  $\sigma = 10\%$ ,  $r_{\$} = 6\%$ ,  $r_e = 3\%$ ,  $T = 0.5$  yr.

Strike ( $K$ )	Vanilla put	Down-and-In		Up-and-Out		
		$H = 0.80$	$H = 0.85$	$H = 0.95$	$H = 1.00$	$H = 1.05$
$K = 0.80$	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007
$K = 0.90$	0.0188	0.0066	0.0167	0.0174	0.0188	0.0188
$K = 1.00$	0.0870	0.0134	0.0501	0.0633	0.0847	0.0869

### Reading the table:

- Vanilla put is always the most expensive (no path restriction).
- Down-and-in approaches vanilla as  $H \uparrow K$  (easier to trigger).
- Up-and-out approaches vanilla as  $H \rightarrow \infty$  (knockout hard to trigger).

## Lookback Options

- **Floating-strike Lookback Call.** Buy at the *minimum* price seen.

$$\text{Payoff} = S_T - S_{\min}.$$

- **Floating-strike Lookback Put.** Sell at the *maximum* price seen.

$$\text{Payoff} = S_{\max} - S_T.$$

- **Fixed-strike Lookback Call.**  $\max(S_{\max} - K, 0)$ .
- **Fixed-strike Lookback Put.**  $\max(K - S_{\min}, 0)$ .
- **Cost intuition.** Lookbacks are *expensive* — typically  $1.5\text{--}2.0\times$  the vanilla cost at ATM, depending on  $\sigma\sqrt{T}$ .  
You're being granted perfect hindsight; the seller charges for the privilege.
- **Pricing.** Closed-form available under continuous monitoring + GBM (Goldman-Sosin-Gatto 1979). Discrete monitoring needs adjustment.
- **Related: shout option.** Holder can “shout” once during the life to lock in the intrinsic value. Final payoff =  $\max(\text{locked-in value}, \text{European payoff})$ .

# Lookback Options

## Path-Dependent Exotic Options with Path-Dependent Valuation

- **Comments**

- Lookback options are usually **expensive** because the holder receives perfect hindsight on the price path.
- Closed-form pricing formulas exist under:
  - continuous monitoring of the underlying price,
  - lognormal (GBM) price dynamics.
- Related to *shout options*: the holder can “freeze” the intrinsic value once during the option’s life.
  - A shout option lets the holder lock in the intrinsic value at one chosen time. Final payoff is the maximum of the locked-in value and the European payoff at maturity.

## Non-Standard American: Bermudan Options

- **Bermudan options:** exercisable only on a *specific set* of pre-specified dates (between European and American).
- **Often have** time-varying strike or other features that adjust on a schedule.
- **Pricing.** Same techniques as American (binomial/trinomial trees with backward induction), but exercise checked only on Bermudan nodes.
- **Applications.**
  - Callable bonds (call rights only on coupon dates).
  - Korean ELS: *autocall* dates are Bermudan early-redemption rights.
  - Employee stock options with reset/ratchet features.
- **Example.** A 7-year warrant exercisable only on specified dates in years 3–7, with strike schedule:
  - Years 3–4:  $K = \$30$
  - Years 5–6:  $K = \$32$
  - Year 7:  $K = \$33$

Reflects the firm's desire to control dilution timing.

# Path-Dependent with Path-Dependent Valuation

Asian Options

# Asian Options

- **Definition.** Payoff based on the *average* price of the underlying over a specified period (arithmetic or geometric; continuous or discrete monitoring).
- **Why average?**
  - Economic exposure naturally tied to an average (e.g., monthly USD revenue from EUR sales, energy procurement, commodity stockpiles).
  - Hard to manipulate a window-average price; easier to manipulate a single closing print.
  - Cheap protection when point-in-time prices are noisy.
  - Some convertible bonds embed Asian features (e.g., 20-day average near maturity).
- **Valuation insight.**
  - Averaging reduces effective volatility of the payoff vs. using  $S_T$ .
  - Lower volatility  $\Rightarrow$  **Asian options are cheaper than European** (same strike).

# Basic Types of Asian Options

- **Averaging methods.** Sampling every  $h$  time units from 0 to  $T$ :
  - Arithmetic average:  $A(T) = \frac{1}{N} \sum_{i=1}^N S_{ih}$
  - Geometric average:  $G(T) = (\prod_{i=1}^N S_{ih})^{1/N}$
  - $G(T) \leq A(T)$  always  $\Rightarrow$  geometric Asians are cheaper.
- **Average-price vs. Average-strike.** Either the underlying or the strike can be averaged.

	<b>Arithmetic</b>	<b>Geometric</b>
Average price call	$\max[0, A(T) - K]$	$\max[0, G(T) - K]$
Average price put	$\max[0, K - A(T)]$	$\max[0, K - G(T)]$
Average strike call	$\max[0, S_T - A(T)]$	$\max[0, S_T - G(T)]$
Average strike put	$\max[0, A(T) - S_T]$	$\max[0, G(T) - S_T]$

# Example: Hedging Currency Exposure

Why Asian options fit a year of monthly FX revenue

- XYZ receives €100m monthly. Let  $x_i$  be the spot USD/EUR rate in month  $i$ . Year-end USD value (ignoring interest):

$$\text{€ } 100\text{m} \times \sum_{i=1}^{12} x_i = \text{€ } 100\text{m} \times 12 \cdot \underbrace{\left( \frac{1}{12} \sum_{i=1}^{12} x_i \right)}_{\text{arithmetic average}}.$$

- Exposure is proportional to the **arithmetic average exchange rate** over the year, not the year-end rate alone.
- Natural hedge: **arithmetic average-price put** with floor  $K$ :

$$\max\left(0, K - \frac{1}{12} \sum_{i=1}^{12} x_i\right).$$

- This protects XYZ against a year-long weakening of the euro — not just a single bad day.

## Example: Hedging Strategy Comparison

- Parameters: spot \$0.90/EUR,  $K = 0.90$ ,  $r_s = 6\%$ ,  $r_e = 3\%$ ,  $\sigma = 10\%$ ,  $T = 1$  yr.
- Total cost of hedging the *full* year of monthly exposure:

Hedge	Total cost (\$ per € 1 of monthly notional)
(a) 12 European puts, all expiring at $T = 1$	0.2753
(b) Basket: one monthly put per cash flow	0.2178
(c) 12 geometric-average puts	0.1796
(d) 12 arithmetic-average puts	0.1764
(e) Currency forward (no premium)	—

- (a) > (b): longest-dated options carry the most time value.
- (b) > (c),(d): averaging reduces effective volatility.
- (d) < (c): geometric average  $\leq$  arithmetic, so geometric put pays more often  $\Rightarrow$  pricier.
- Forward kills FX risk for zero premium — but also kills upside.

# Pricing Asian Options

- **Geometric average: closed form.**

Under GBM, the geometric average is itself lognormal. Apply a Black-style formula with analytically derived mean and variance of the average. (Useful as a benchmark and control variate.)

- **Arithmetic average: no closed form.**

- Arithmetic average of lognormals is *not* lognormal.
- Trees: possible but the average is a separate state variable — tree non-recombining, computationally heavy.
- **Monte Carlo**: most common; handles path dependence naturally.
- Variance reduction: **geometric Asian as a control variate**, antithetic sampling, quasi-Monte Carlo.

- **Approximation methods.** Moment matching, lognormal approximations, PDE methods.

# Exchange Options

## Multivariate Options

- **Definition**

- An exchange option pays off only if one asset outperforms another.

$$\max(0, S_T - N_T)$$

- Useful when the goal is *relative* performance hedging (e.g., commodity spreads, equity relative value, FX crosses).

- **Value of a European Exchange Call (Margrabe Formula)**

$$Se^{-q_S T} N(d_1) - Ne^{-q_N T} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{N}\right) + (q_N - q_S + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and the **effective volatility** of the spread is

$$\sigma = \sqrt{\sigma_S^2 + \sigma_N^2 - 2\rho\sigma_S\sigma_N}.$$

# Multivariate Exotics

Exchange and Quanto

# Exchange Options

- **Definition.** Pays off only if one asset outperforms another:

$$\max(0, S_T - N_T).$$

Used for relative-value hedging — commodity spreads (crude vs. heating oil), equity pair trades, FX crosses.

- **Margrabe formula (1978).** European exchange call:

$$Se^{-q_S T} N(d_1) - Ne^{-q_N T} N(d_2),$$
$$d_1 = \frac{\ln(S/N) + (q_N - q_S + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$
$$\sigma = \sqrt{\sigma_S^2 + \sigma_N^2 - 2\rho\sigma_S\sigma_N}.$$

- **Intuition.** It's the BSM formula applied to the ratio  $S/N$ . The effective volatility falls with positive correlation  $\rho \Rightarrow$  pricier exchange option when assets move together.
- **Use case: implied correlation.** Given the exchange-option price, back out  $\rho$ . Widely used in multi-asset desks, basket products, FX.

# Quanto Options

- **Quanto** = “quantity-adjusting” option.  
Investor holds an asset denominated in one currency but receives the payoff in another currency, at a **fixed** (or effectively hedged) exchange rate.
- **Why useful.** Strips FX risk from a foreign-asset bet.
- **Classic example.** Nikkei put warrants traded on AMEX:
  - Payoff in USD,
  - Linked to the yen price of the Nikkei vs. a yen strike.
  - Investor gains Japanese equity exposure without USD/JPY risk.
- **Hedging subtlety.** A quanto embeds a currency forward whose notional varies with the underlying. A simple FX forward cannot perfectly hedge a quanto — hence the name “quantity-adjusting.”
- **Korean parallel.** Many Korean retail products (S&P 500–linked ELS, US-stock ELS) are quanto-style: payoffs paid in KRW at a fixed FX rate.

# Quanto vs. Regular Foreign Call

## Numerical Illustration

- Foreign call: strike  $K = 100$  (foreign), payoff  $\max(S_T - K, 0)$  in foreign currency.
- Spot FX (domestic per foreign):  $X_0 = 1.0$ .
- Two FX scenarios at maturity: weak ( $X_T = 0.8$ ), strong ( $X_T = 1.2$ ).

$S_T$	$\max(S_T - K, 0)$ (foreign)	Regular (weak FX) $X_T = 0.8$	Regular (strong FX) $X_T = 1.2$	Quanto (fixed $X_0 = 1.0$ )
80	0	0	0	0
100	0	0	0	0
120	20	16	24	20
140	40	32	48	40

Regular call payoff =  $X_T \cdot \max(S_T - K, 0)$  — FX-dependent.

Quanto payoff =  $X_0 \cdot \max(S_T - K, 0)$  — **independent of  $X_T$** .

# Pricing Exotic Options

Trees & Monte Carlo

# Pricing Approaches

- **Modified BSM formulas.**

Closed (or semi-closed) form for certain structures: geometric Asian, barriers, exchange. Require lognormality + continuous monitoring (see Hull §26).

- **Binomial / trinomial trees.**

Best for early-exercise features (American, Bermudan). Path dependence grows the tree fast — non-recombining for arithmetic averages.

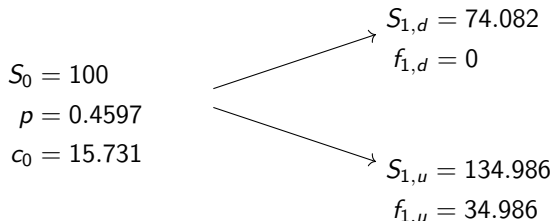
- **Monte Carlo simulation.**

The most flexible. Handles path dependence, multiple underlyings, complex triggers, and general dynamics. Convergence  $O(1/\sqrt{N})$ ; use variance reduction.

# Risk-Neutral Binomial Tree

- One step:  $S_0 = 100$ ,  $K = 100$ ,  $T = 1$ ,  $r = 2\%$ ,  $\sigma = 30\%$ .
- Up:  $u = e^{\sigma\sqrt{T}} = 1.34986$ , Down:  $d = 1/u = 0.74082$ .
- RN prob:  $p = \frac{e^{rT} - d}{u - d} = 0.4597$ .
- Derivative with payoff  $f(S_1)$ :

$$V_0 = e^{-rT} [p f(S_{1,u}) + (1 - p) f(S_{1,d})].$$



# Monte Carlo on the Risk-Neutral Tree

- Instead of computing the expectation analytically, **simulate** the up/down draws.
- Excel: `RAND()` gives  $U \sim \text{Uniform}(0, 1)$ .
- ① Draw  $N$  values of `RAND()`.
  - If `RAND()`  $< p$ : up state. Else: down state.
- ② For each simulation  $j$ , obtain  $S_1^{(j)} \in \{S_{1,u}, S_{1,d}\}$ .
- ③ Compute payoff  $V(S_1^{(j)}) = \max(S_1^{(j)} - K, 0)$ .
- ④ Monte Carlo estimator:

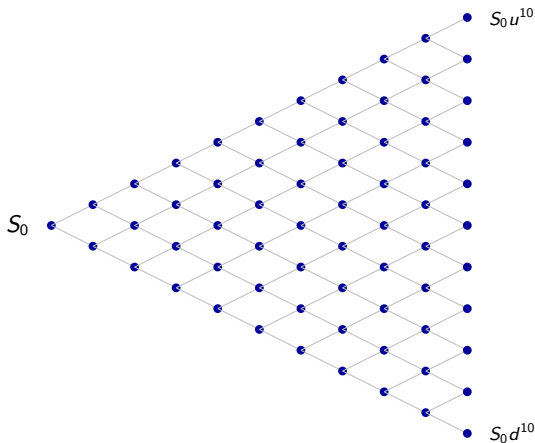
$$\hat{V}_0 = \frac{1}{N} \sum_{j=1}^N e^{-rT} V(S_1^{(j)}).$$

## Monte Carlo: Small-Sample Illustration ( $N = 10$ )

RAND()	Move	$S_T$	Payoff	Discounted
0.4573	up	134.986	34.986	34.293
0.3939	up	134.986	34.986	34.293
0.0901	up	134.986	34.986	34.293
0.8781	down	74.082	0	0
0.6587	down	74.082	0	0
0.7596	down	74.082	0	0
0.7980	down	74.082	0	0
0.0617	up	134.986	34.986	34.293
0.9692	down	74.082	0	0
0.3927	up	134.986	34.986	34.293
<b>Average</b>				17.147
<b>Std. error</b>				5.715

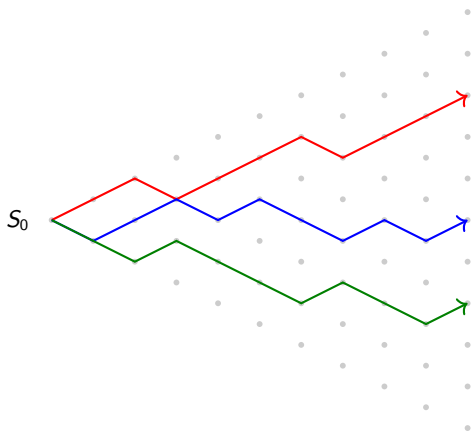
- With  $N = 10$ :  $\hat{V}_0 = 17.147$  vs. true  $V_0 = 15.731$ . Wide CI.
- Increase  $N$  to tighten precision (rate  $\propto 1/\sqrt{N}$ ).

# Multi-Step Binomial Tree (10 steps)



- As the number of steps increases, the tree-implied price converges to the BSM price.
- Each path probability:  $\binom{n}{k} p^k (1-p)^{n-k}$ .
- Recombines  $\Rightarrow$  only  $n + 1$  terminal nodes, not  $2^n$  — tractable.

## Monte Carlo on the Multi-Step Tree



- Generate stock paths: if  $\text{RAND}() < p$ , up; else down.
- Simulate 1,000+ such paths, compute  $f_T e^{-rT}$ , take the average.
- Path memory only matters if the payoff is path-dependent (e.g., Asian, barrier, lookback).

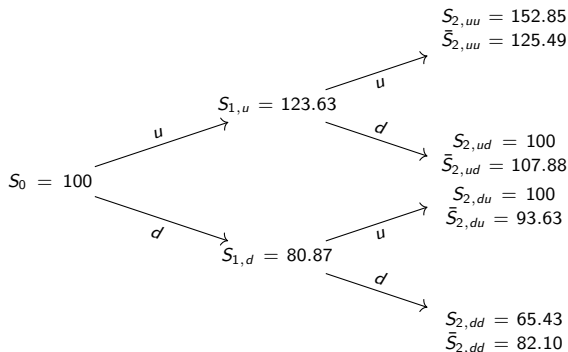
# Why Monte Carlo, When We Have Trees?

- **Trees** work well when the payoff depends *only* on the terminal price  $S_T$ .
- **Path-dependent payoffs** depend on the entire price sequence:
  - Asian, lookback, barrier (with continuous monitoring), autocallable ELS.
- Number of distinct paths grows as  $2^n \Rightarrow$  tree becomes computationally infeasible.
- **Example: Asian call.**

$$\max \left( \frac{1}{T} \sum_{t=1}^T S_t - K, 0 \right).$$

- Paths ending at the same  $S_T$  can have very different averages.
  - Knowing the terminal node alone is not enough.
- **Monte Carlo handles path dependence naturally** by simulating sample paths instead of enumerating the tree.

# Why Trees Fail for Path-Dependent Payoffs



- Even though  $S_{2,ud} = S_{2,du} = 100$ , the path **averages differ**:  $\bar{S}_{2,ud} \neq \bar{S}_{2,du}$ .
- Tree becomes **non-recombining**:  $2^n$  nodes vs.  $n + 1$  for vanilla.
- Monte Carlo simulates paths individually — avoiding the exponential blowup.

# Monte Carlo Without Trees

- **Monte Carlo is not restricted to binomial trees.**  
Trees were just a teaching device for risk-neutral pricing.
- **Key requirement.** The price process must admit risk-neutral pricing (no-arbitrage + dynamic replication).
- **Once risk-neutrality holds**, you can simulate *any* valid process:
  - Lognormal GBM (Black-Scholes-Merton);
  - Stochastic volatility (Heston, SABR);
  - Stochastic interest rates;
  - Jump-diffusion (Merton, Kou).
- This makes MC the workhorse for pricing under realistic dynamics that lack closed forms.

# Monte Carlo Under Lognormality (GBM)

- GBM under  $Q$ :  $dS_t = rS_t dt + \sigma S_t dW_t$ .
- Discretized:

$$S_{t+\Delta t} = S_t \exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma \varepsilon_t \sqrt{\Delta t}\right), \quad \varepsilon_t \sim N(0, 1).$$

- Equivalently:

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t, \sigma^2\Delta t\right).$$

- **Algorithm.**

- 1 Draw  $\varepsilon_t \sim N(0, 1)$  (in Excel: `NORM.S.INV(RAND())`).
- 2 Update  $S_{t+\Delta t}$ .
- 3 Repeat over time steps to build a full path.
- 4 Average discounted payoffs across many paths.

# Monte Carlo with Correlated Factors

- Option on the *max of two stocks* (e.g., Google  $S$  and Apple  $N$ ):

$$\max\left(\frac{S_T}{S_0}, \frac{N_T}{N_0}\right).$$

- Under risk-neutral GBM:

$$S_{t+\Delta t} = S_t \exp\left(\left(r - \frac{1}{2}\sigma_S^2\right)\Delta t + \sigma_S \varepsilon_{1,t} \sqrt{\Delta t}\right),$$
$$N_{t+\Delta t} = N_t \exp\left(\left(r - \frac{1}{2}\sigma_N^2\right)\Delta t + \sigma_N \varepsilon_{2,t} \sqrt{\Delta t}\right).$$

- Correlated shocks** (Cholesky for  $2 \times 2$ ):

$$\varepsilon_{2,t} = \rho \varepsilon_{1,t} + \sqrt{1 - \rho^2} v_t, \quad v_t \sim N(0, 1) \text{ indep.}$$

- For higher dimensions, replace with full Cholesky factorization.

## Multi-Factor Monte Carlo Example

- Parameters:  $\sigma_S = \sigma_N = 0.30$ ,  $r = 2\%$ ,  $\rho = 0.7$ ,  $T = 1$ .
- For each path  $i$ , discounted payoff:

$$V^{(i)} = e^{-rT} \max\left(\frac{S_T^{(i)}}{S_0}, \frac{N_T^{(i)}}{N_0}\right).$$

- Estimate  $\hat{V}_0 = \frac{1}{n} \sum_{i=1}^n V^{(i)}$ . Simulation gives  $\hat{V}_0 \approx 1.134$ .
- “Relative return” option:**

$$\max\left(\frac{S_T}{S_0} - \frac{N_T}{N_0}, 0\right).$$

Simulation gives  $\hat{V}_0 \approx 0.10$ . Makes sense: positive correlation reduces the volatility of the spread, so the option is cheaper.

- Korean angle.** Worst-of two-index ELS (KOSPI 200 & S&P 500) requires this exact machinery; correlation  $\rho$  is a key risk parameter.

# Summary

- **Main tools for pricing derivatives:**
  - Modified Black–Scholes–Merton formulas,
  - Binomial/trinomial trees,
  - Monte Carlo simulation.
- **Binomial trees:**
  - Essential for American-style options where optimal early exercise must be evaluated.
  - Useful when backward induction and ordered outcomes are important.
- **Monte Carlo simulation:**
  - Widely used in practice for complex or path-dependent derivatives.
  - Steps:
    - ① Simulate many sample paths under the risk-neutral measure.
    - ② Compute discounted payoffs per path.
    - ③ Estimate value as the average of payoffs.

# Summary

- **Monte Carlo is especially useful for:**
  - Path-dependent payoffs:
    - Asian options, barrier options, lookback options,
    - Options on maxima/minima, rainbow and basket options.
  - Multi-asset derivatives where correlations matter.
- **Also useful for pricing under general dynamics:**
  - Stochastic volatility,
  - Stochastic interest rates,
  - Jump-diffusion processes.
- Increasing computational power has made Monte Carlo one of the most flexible and powerful pricing tools.