BUSS386 Problem Set 12 — Solutions

Volatility

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Problem 1 — Historical, EWMA, GARCH

Prices: 360.0, 363.6, 360.0, 367.2, 363.5, 369.8.

(a) Log returns $R_t = \ln(S_t/S_{t-1})$:

t	1	2	3	4	5
R_t	0.00995	−0.00995	0.01980	−0.01012	0.01718
R_t^2	0.0000990	0.0000990	0.000392	0.000102	0.000295

$\sum R_t^2 = 0.000988$.

(b) $\hat{\sigma}^2_{\text{daily}} = 0.000988/5 = \boxed{0.0001976}$.
$\hat{\sigma}_{\text{daily}} = 0.01406 \Rightarrow \hat{\sigma}_{\text{annual}} = 0.01406\sqrt{252} = \boxed{22.3\%}$.

(c) EWMA with $w = 0.94$. Reorder from most recent (k=0=R_5) to oldest:

k	0	1	2	3	4
R^2	0.000295	0.000102	0.000392	0.0000990	0.0000990
w^k	1.0000	0.9400	0.8836	0.8306	0.7807

$\sum w^k R^2 = 0.000898$, $\sum w^k = 4.4349$.
$\hat{\sigma}^2_{\text{daily}} = 0.000898/4.4349 = 0.0002024$, $\hat{\sigma}_{\text{annual}} = \sqrt{0.0002024 \times 252} = \boxed{22.6\%}$.

(d) GARCH: $\sigma^2_\infty = 0.0001976$, $\omega = (1 - 0.99)(0.0001976) = 1.976\times10^{-6}$. Start $\sigma_1^2 = \sigma^2_\infty$.
$\epsilon_1 = R_1 = 0.00995$, $\epsilon_1^2 = 9.9\times10^{-5}$:

$$\sigma_2^2 = 1.976\times10^{-6} + 0.05(9.9\times10^{-5}) + 0.94(0.0001976) = \boxed{0.0001927}, \quad \sigma_2 = \boxed{1.39\%/\text{day}}.$$

$\epsilon_2 = R_2 = -0.00995$, $\epsilon_2^2 = 9.9\times10^{-5}$:

$$\sigma_3^2 = 1.976\times10^{-6} + 0.05(9.9\times10^{-5}) + 0.94(0.0001927) = \boxed{0.0001881}, \quad \sigma_3 = \boxed{1.37\%/\text{day}}.$$

Problem 2 — Implied volatility inversion

$S_0 = K = 100$, $T = 0.25$, $r = 0.04$, $c_{\text{mkt}} = 5.00$.

(a) $\sigma = 0.20$: $\sigma\sqrt{T} = 0.10$, $d_1 = (0 + 0.06 \cdot 0.25)/0.10 = 0.15$, $d_2 = 0.05$.
$N(0.15) = 0.5596$, $N(0.05) = 0.5199$.
$c = 100(0.5596) - 100e^{-0.01}(0.5199) = 55.96 - 51.48 = \boxed{\$4.48}$.
Market \$5.00 > \$4.48 ⇒ IV is *above* 20%.

(b) $\sigma = 0.25$: $\sigma\sqrt{T} = 0.125$, $d_1 = (0 + 0.07125 \cdot 0.25)/0.125 = 0.1425$, $d_2 = 0.0175$.
$N(0.1425) = 0.5567$, $N(0.0175) = 0.5070$.
$c = 100(0.5567) - 100e^{-0.01}(0.5070) = 55.67 - 50.19 = \boxed{\$5.48}$.
Market \$5.00 is *between* \$4.48 (at 20%) and \$5.48 (at 25%) ⇒ IV is in $(20\%, 25\%)$.

(c) Linear interpolation:

$$\sigma_{\text{imp}} \approx 0.20 + \frac{5.00 - 4.48}{5.48 - 4.48}(0.25 - 0.20) = 0.20 + \frac{0.52}{1.00}(0.05) = 0.20 + 0.026 = \boxed{22.5\%}.$$

A few Newton iterations would refine this slightly but 22.5% is within the bid-ask of typical quotes.

Problem 3 — Vol risk premium and gamma scalping

(a) $\sigma^2_{\text{real}} - \sigma^2_{\text{imp}} = 0.15^2 - 0.19^2 = 0.0225 - 0.0361 = -0.0136 < 0$.
A long-gamma scalper has $\boxed{\text{negative expected P\&L}}$ — on average, realized vol does not cover the implied vol you paid.

(b) Short vol = collect the premium that long-vol pays. The volatility risk premium of ∼4 pp is structural: investors pay to hedge tail risk. JEPI / covered-call ETFs / Korean ELS issuers all sell this insurance.

(c) The catch: short-vol strategies lose catastrophically when realized vol *exceeds* implied — exactly when you can least afford it (1987, 2008, 2018 XIV, 2020 COVID, 2024 Aug 5 KOSPI 200). The carry is real; so is the crash.

Problem 4 — Smile-adjusted equity put

$S_0 = 360$, $K = 342$, $T = 0.5$, $r = 0.03$, $q = 0.015$.
$\ln(S_0/K) = \ln(360/342) = 0.05129$. $(r - q)T = 0.0075$.

(a) $\sigma = 0.18$: $\sigma\sqrt{T} = 0.1273$.
$d_1 = (0.05129 + (0.0075 + 0.0162)(0.5))/0.1273 = (0.05129 + 0.01185)/0.1273$
wait: $(r - q + \sigma^2/2)T = (0.0075 + 0.5 \cdot 0.0324) \cdot 0.5 = (0.0075 + 0.0162) \cdot 0.5 = 0.01185$
$d_1 = (0.05129 + 0.01185)/0.1273 = 0.06314/0.1273 = 0.4960$.
$d_2 = 0.4960 - 0.1273 = 0.3687$.
$N(-0.4960) = 0.3099$, $N(-0.3687) = 0.3562$.
$p = Ke^{-rT}N(-d_2) - S_0e^{-qT}N(-d_1) = 342(0.9851)(0.3562) - 360(0.9925)(0.3099)$
$= 119.99 - 110.71 = \boxed{9.28}$ pts.

(b) $\sigma = 0.21$: $\sigma\sqrt{T} = 0.1485$.
$(r - q + \sigma^2/2)T = (0.0075 + 0.5 \cdot 0.0441)(0.5) = (0.0075 + 0.02205)(0.5) = 0.01478$
$d_1 = (0.05129 + 0.01478)/0.1485 = 0.06607/0.1485 = 0.4449$.
$d_2 = 0.4449 - 0.1485 = 0.2964$.

$N(-0.4449) = 0.3282$, $N(-0.2964) = 0.3834$.
$p = 342(0.9851)(0.3834) - 360(0.9925)(0.3282) = 129.16 - 117.28 = \boxed{11.88}$ pts.

(c) Smile premium $= 11.88 - 9.28 = \boxed{2.60}$ pts. Per contract: $2.60\times$ ₩250,000 $=$ $\boxed{\text{₩650,000}}$ extra cost.

Problem 5 — Surface interpolation

(a) Read directly: $\sigma(K/S_0 = 1.05, T = 6\text{mo}) = \boxed{16.0\%}$.

(b) At $K/S_0 = 1.05$, $T = 6\text{mo} \to 16.0\%$, $T = 12\text{mo} \to 16.5\%$. Linear at $T = 9$ months:

$$\sigma = 16.0\% + \frac{9-6}{12-6}(16.5\% - 16.0\%) = 16.0\% + 0.25\% = \boxed{16.25\%}.$$

(c) At $T = 6\text{mo}$, $K/S_0 = 1.00 \to 17.5\%$, $K/S_0 = 1.05 \to 16.0\%$. Linear at $K/S_0 = 1.03$:

$$\sigma = 17.5\% + \frac{0.03}{0.05}(16.0\% - 17.5\%) = 17.5\% - 0.9\% = \boxed{16.6\%}.$$

(Pure linear in K here because the target $T = 6\text{mo}$ is on the grid; bilinear in general.)

(d) At $T = 3\text{mo}$: skew from 21% (0.95) down to 15% (1.10), a drop of 6 pp. At $T = 12\text{mo}$: 19% to 16.5%, a drop of only 2.5 pp. So skew is $\boxed{\text{steeper at } T = 3 \text{ months}}$.
Typical because short-dated options price imminent event risk (the next earnings / FOMC / Korean MPC); long-dated options price the long-run RN distribution where event risk averages out.

Problem 6 — Variance swap

$K_{\text{var}} = 0.0324$, Notional = ₩1B.

(a) Realized variance = 0.0400:
Payoff $= 1\text{B} \times (0.0400 - 0.0324) = 1\text{B} \times 0.0076 = \boxed{+\text{₩}7.6\text{M}}$ (gain for the buyer).

(b) Realized variance = 0.0225:
Payoff $= 1\text{B} \times (0.0225 - 0.0324) = -1\text{B} \times 0.0099 = \boxed{-\text{₩}9.9\text{M}}$ (loss for the buyer).

(c) Variance swaps are exactly replicable by a $1/K^2$-weighted strip of OTM options (Carr-Madan); volatility swaps involve $\sqrt{\text{Var}}$ which is concave and not perfectly replicable, so variance is the cleaner product for dealers to make markets in.