Option Greeks

BUSS386. Futures and Options

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Lecture Outline

- The five Greeks: $\Delta, \Gamma, \Theta, \nu, \rho$
- Delta hedging and the market-maker's job
- Gamma scalping: long vol vs short vol (simplified)
- Delta-gamma, vega hedging; the Θ–Γ identity
- Applications: principal-protected notes, portfolio insurance

Disclaimer. All formulas below are BSM-model Greeks. The *actual* sensitivities in the market usually differ from what BSM predicts — that gap (model risk) is a theme of this lecture.

"The Greeks" — Sensitivities of the Option Price

The Greeks measure how an option's value V responds to each input.

Greek	Symbol	Definition	Measures
Delta	Δ	$\partial V/\partial S$	directional (market) risk
Gamma	Γ	$\partial^2 V/\partial S^2$	curvature / hedge stability
Theta	Θ	$\partial V/\partial t$	time decay
Vega	ν	$\partial V/\partial \sigma$	volatility risk
Rho	ρ	$\partial V/\partial r$	interest-rate risk

- Used for both **risk management** (know your exposures) and **trading** (bet on one Greek, hedge the others).
- Under BSM they have clean closed forms (this lecture).
- Portfolio Greeks **add up**: $\Delta_\Pi = \sum_i \Delta_i$, etc.

Delta, Δ

1. Delta: Sensitivity of option to changes in the underlying price.

$$\Delta = \frac{\partial V}{\partial S} = N(d_1) \text{ for Calls}$$

  - For dividend paying underlyings: $e^{-q(T-t)}N(d_1)$
  - For put: $N(d_1) - 1$. With dividends: $e^{-q(T-t)}(N(d_1) - 1)$

- It tells how many units of the underlying asset one should trade in order to hedge the market risk exposure of the option.
  - For example, if $\Delta = 0.50$ for a given call option, the position that is long one call and short 0.50 shares of stock will be hedged against a (small) change in the stock price up or down (Delta neutral hedge)
  - Delta measures **market risk**.

- Approximately the probability that an option finishes in the money (in a risk neutral world).
  - $\text{Prob}\,(S_T \geq K) = N(d_2) = N(d_1 - \sigma\sqrt{T}) \propto N(d_1)$

Delta, Δ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

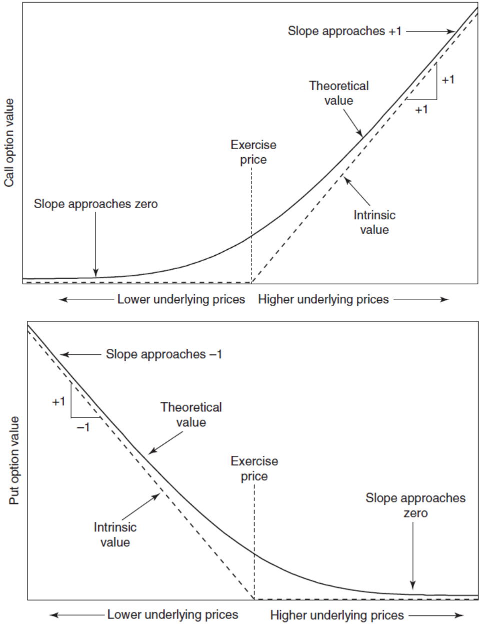
$$d_1 = \frac{ln(49/50) + (0.05 + 0.2^2/2)0.3846}{0.2 \times \sqrt{0.3846}} = 0.0542$$

- Delta is $N(d_1) = 0.522$. When the stock price changes by ΔS, the option price changes by $0.522\Delta S$.

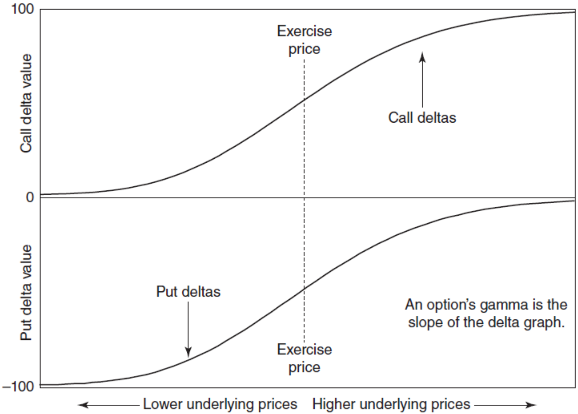
Delta, Δ (cont'd)

- The delta of is positive for calls and negative for puts.
- The delta is close to ± 1 for deep in the money options.
- The delta of far out of the money option is close to 0.
- At the money option has delta of about ± 0.50.

Delta, Δ (cont'd)



Delta, Δ (cont'd)



Same shape for call and put: the put-call parity!

Gamma, Γ

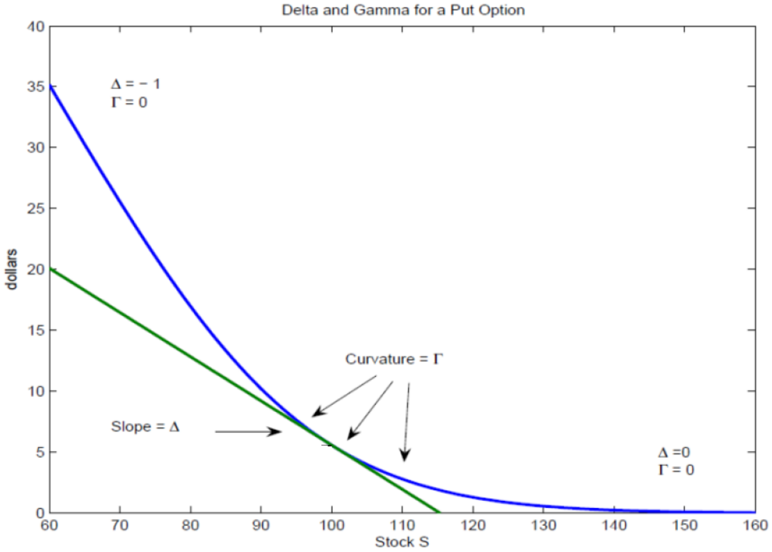
2. Gamma: Sensitivity of Delta to changes in the underlying price.

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$

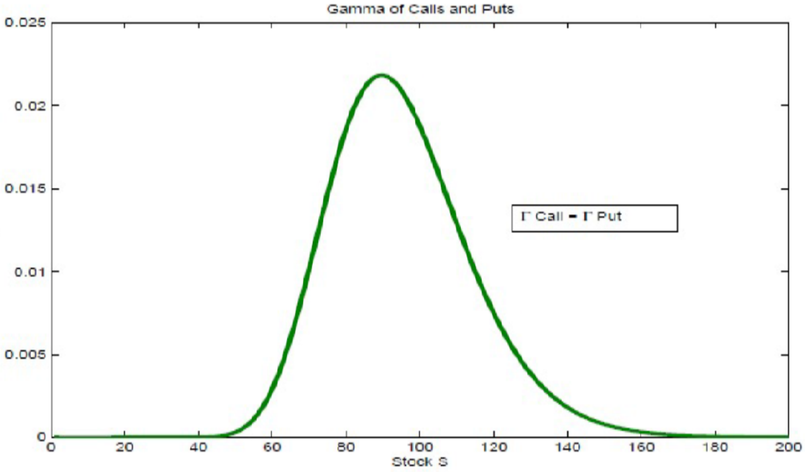
where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

- Identical for both calls and puts. For dividend paying underlyings?
- Gamma measures risk for a delta neutral hedge.
- Gamma is related to the curvature of the option value function.
  - For a long position, always positive.
  - Gamma is the largest at the money.
  - Gamma is small in the deep in or out of the money.

Gamma, Γ (cont'd)

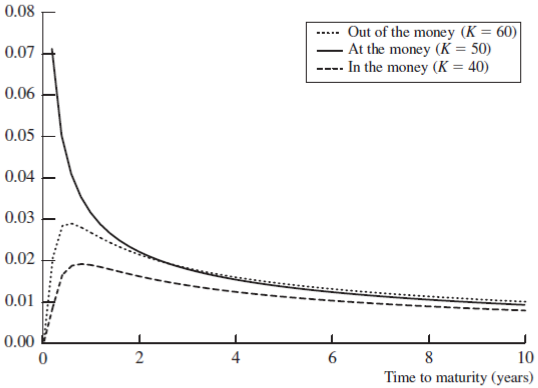


Gamma, Γ (cont'd)



Gamma, Γ (cont'd)

Variation of gamma with time to maturity for a stock option ($S = 50, r = 0, \sigma = 25\%$).



For an at-the-money option, gamma increases as the time to maturity decreases. Short-life at-the-money options have very high gammas, which means that the value of the option holder's position is highly sensitive to jumps in the stock price.

Gamma, Γ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$\frac{N'(d_1)}{S\sigma\sqrt{T}} = 0.066$$

- When the stock price changes by ΔS, the delta of the option changes by $0.066\Delta S$.

Gamma, Γ (cont'd)

- A call has a Delta of 0.54 and Gamma of 0.04.
  - Stock goes up $1: Delta will become more positive by the Gamma amount.
  - New Delta value: 0.58

- Another call has a Delta of 0.75 and Gamma of 0.03
  - Stock is down $1: Delta will become less positive by Gamma amount.
  - New Delta value: 0.72

- XYZ: $S = \$50, K = \$50, C = \$2, \Delta = 0.50, \Gamma = 0.06$
  - Should XYZ go up to $51, the 50 strike call will be worth around $2.50 when using delta only.
  - Using gamma as well: $c(51) - c(50) = \Delta(51 - 50) + \frac{1}{2}\Gamma(51 - 50)^2$
  - Delta $\approx$ (Dollar) duration, Gamma $\approx$ (Dollar) convexity

Theta, Θ

3. Theta: Sensitivity of option to passage of time, t.

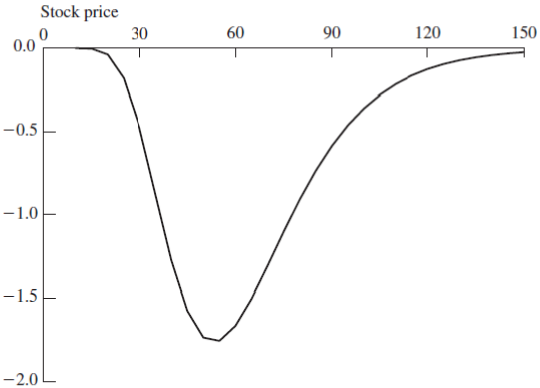
$$\Theta = \frac{\partial V}{\partial t} = \begin{cases} -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) & \text{for Calls} \\ -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) & \text{for Puts} \end{cases}$$

where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

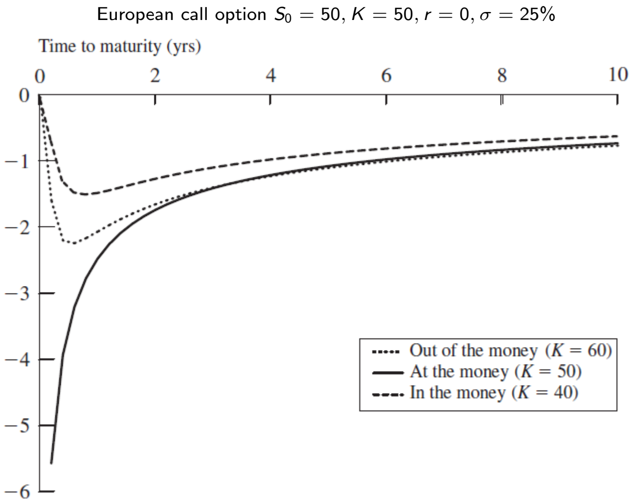
- Theta measures **Time Decay**. Options lose value as time passes (holding all else constant).
- Theta decreases (more negative) when the option closer to expiration and at the money.
- Deep ITM or deep OTM options have low theta because little time value remains.

Theta, Θ (cont'd)

European call option $K = 50, r = 0, \sigma = 25\%, T = 2$



Theta, Θ (cont'd)



Theta (Θ) (Cont'd)

(Assume long option positions throughout.)

- **Call on a non-dividend-paying stock:** $\Theta < 0$
  - As time passes (with S fixed), the **variance of S_T shrinks**. A narrower distribution reduces the value of optionality.
  - The strike price K is like a **debt due at maturity**. As time passes, the discount factor $e^{-r(T-t)}$ becomes smaller $\rightarrow$ the present value of the "debt" rises. This hurts a long call holder.

- **Call on a dividend-paying stock:** Θ **can be positive**
  - A call holder does not receive dividends. If a dividend is paid soon, the stock price drops on ex-div date $\rightarrow$ bad for the call.
  - As time passes and the dividend is avoided (or gets closer to being avoided), the call's relative value can **increase**, making $\Theta > 0$ possible.
  - Deep OTM calls have very low sensitivity to this effect $\rightarrow$ Theta remains near zero.

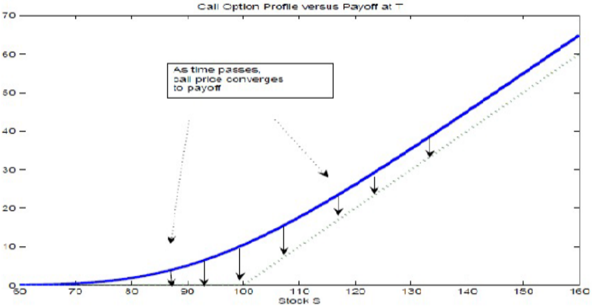
Theta (Θ) (Cont'd)

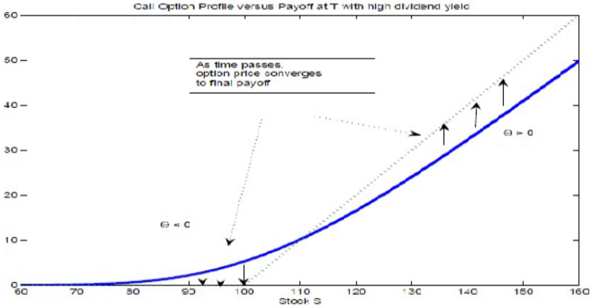
(Assume long option positions throughout.)

- **Put options: Θ can be positive or negative**
  - When S is high (put is OTM): payoff at maturity is likely zero but the put currently has time value. As time passes (with S fixed), this time value decays $\rightarrow \Theta < 0$.
  - When S is very low (deep ITM): payoff is approximately K with probability near 1. The present value of K is $Ke^{-r(T-t)}$, which **increases** as t increases (less discounting). Hence $\Theta > 0$.

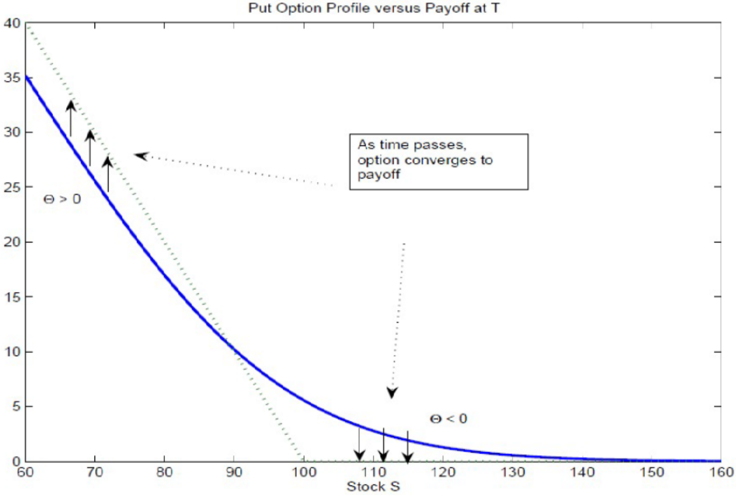
- **American options: Θ is typically negative**
  - Early exercise rights add value, but this value also declines as time passes.
  - The option still loses time value overall $\rightarrow$ **Theta is almost always negative**.

Theta, Θ (cont'd)





Theta, Θ (cont'd)



Theta, Θ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$-\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) = -4.31$$

- The theta is $-4.31/365 = -0.0118$ per calendar day, or $-4.31/252 = -0.0171$ per trading day.

Vega, ν

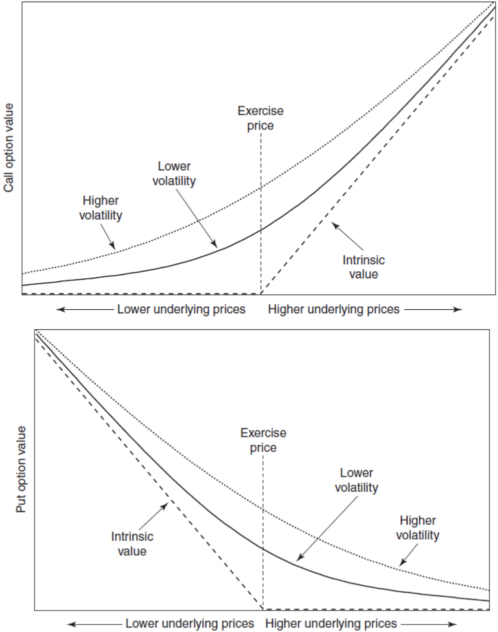
4. Vega: Sensitivity of option to a change in volatility σ.

$$\nu = \frac{\partial V}{\partial \sigma} = S\sqrt{T}N'(d_1) > 0$$

where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

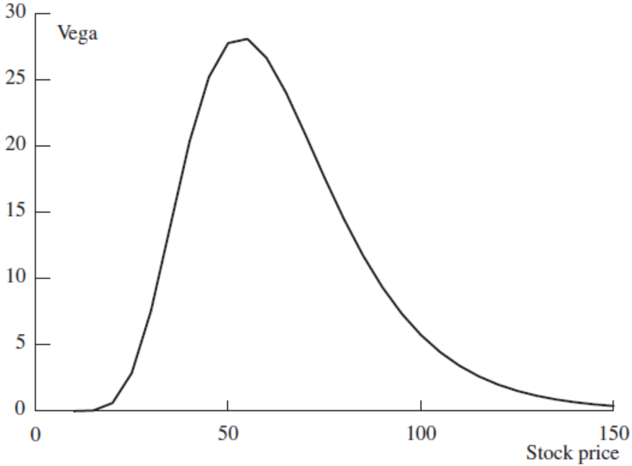
- Vega measures exposure to **Volatility Risk**
- The vega of European and American calls and puts is positive.
- For very deep OTM or ITM options, the vega is close to zero.
- The vega of a call or put peaks near the money.
- Buying a portfolio with positive vega is "buying volatility". Typically we do this by buying a call and a put — a straddle.

Vega, ν (cont'd)



Vega, ν (cont'd)

Variation of vega with stock price for an option $K = 50$, $r = 0$, $\sigma = 25\%$, $T = 2$



Vega, ν (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$S\sqrt{T}N'(d_1) = 12.1$$

- Thus a 1% (0.01) increase in the implied volatility from (20% to 21%) increases the value of the option by approximately $0.01 \times 12.1 = 0.121$.

Rho, ρ

5. Rho: Sensitivity of option to a change in the interest rate.

$$\rho = \frac{\partial V}{\partial r} = \begin{cases} KTe^{-rT}N(d_2) > 0 & \text{for Calls} \\ -KTe^{-rT}N(-d_2) < 0 & \text{for Puts} \end{cases}$$

- Rho measures exposure to Interest Rate Risk.
- It depends on whether the option holder will pay K (call) or receive K (put). The PV of K declines as r increases, making the payment made smaller for the long call and payment received smaller for the long put.

Rho, ρ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$KTe^{-rT}N(d_2) = 8.91$$

- This means that a 1% (0.01) increase in the risk-free rate (from 5% to 6%) increases the value of the option by approximately $0.01 \times 8.91 = 0.0891$.

Greeks at a Glance — Signs for Long Positions

Greek	Long call	Long put	Peaks / largest when
Δ	$+$ (0 to 1)	$-$ (-1 to 0)	deep ITM $\to \pm 1$
Γ	$+$	$+$	ATM, short-dated
Θ	$-$ (usu.)	$-$ (usu.)	ATM, short-dated
ν	$+$	$+$	ATM, long-dated
ρ	$+$	$-$	long-dated

- Short positions: flip every sign.
- Γ and ν are always positive for long options — both reward "more uncertainty."
- Γ and Θ have opposite signs (long options): you pay theta to own gamma. We'll see why next.

Applications

A1: Who Cares About the Greeks? — The Market Maker

- A market maker (MM) who sells you an option **immediately delta-hedges** by trading the underlying.
- The MM is *not* betting on direction — delta is hedged away. The MM is left with exposure to **gamma, vega, theta**.
- A MM runs a book of thousands of options across strikes and maturities, and continuously monitors the **net** $\Delta, \Gamma, \nu, \Theta$.
- The MM's edge: collect the bid-ask spread, and earn the volatility risk premium (sell options at implied vol $>$ realized vol on average).

A1: Delta-Hedging Exercise

- Here are the current market prices for XYZ stock and two XYZ options. The Greek letter risk exposures come from the Black-Scholes model. The interest rate is 8% and the implied volatility is 0.25.

	Market price	delta	gamma	vega	theta
XYZ Stock	100	1	0	0	0
XYZ Call 105 strike, 1 month	1.25	0.29	0.047	0.099	−0.044
XYZ Put 95 strike, 1 month	0.83	−0.21	0.039	0.084	−0.030

- You are long 105-strike calls on 100,000 shares. (That is, you have 100,000 call options, each covering one share.)
  1. How would you set up a delta hedge for this position?
  2. What would the overall hedged position be worth? (What is the net cost to set it up?)
  3. What are the Greek letter exposures for the overall position?

A1: Exercise (cont'd)

	Market price	delta	gamma	vega	theta
XYZ Stock	100	1	0	0	0
XYZ Call 105 strike, 1 month	1.25	0.29	0.047	0.099	−0.044
XYZ Put 95 strike, 1 month	0.83	−0.21	0.039	0.084	−0.030

1. Position delta is $100,000 \times 0.29 = 29,000$. Hedge by shorting $29,000$ shares.
2. $$\begin{aligned} Calls &= 100,000 \times 1.25 = 125,000 \\ Stocks &= -29,000 \times 100 = -2,900,000 \\ Total &= -2,775,000 \end{aligned}$$
3. $$\begin{aligned} Delta &= 100,000 \times 0.29 + (-29,000) \times 1 = 0 \\ Gamma &= 100,000 \times 0.047 + (-29,000) \times 0 = 4,700 \\ Vega &= 100,000 \times 0.099 + (-29,000) \times 0 = 9,900 \\ Theta &= 100,000 \times -.044 + (-29,000) \times 0 = -4,400 \end{aligned}$$

A1: Delta-Hedging Exercise — Next Day

- Tomorrow, XYZ stock opens at 95. Here is the new set of option prices and Greek letters.

	Market price	delta	gamma	vega	theta
XYZ Stock	95	1.0	0	0	0
XYZ Call 105 strike, 1 month	0.30	0.10	0.025	0.047	−0.021
XYZ Put 95 strike, 1 month	3.35	−0.46	0.044	0.108	−0.052

4. If you liquidate right now, what would the profit or loss on the hedged position be?
5. If you don't liquidate, what stock trade will you need to do to become delta neutral again?

A1: Exercise (cont'd)

	Market price	delta	gamma	vega	theta
XYZ Stock	95	1.0	0	0	0
XYZ Call 105 strike, 1 month	0.30	0.10	0.025	0.047	−0.021
XYZ Put 95 strike, 1 month	3.35	−0.46	0.044	0.108	−0.052

4. If you unwind at the new prices your profit is:

$$\begin{aligned} \mathit{Calls} &= 100,000 \times (0.30 - 1.25) = -95,000 \\ \mathit{Stocks} &= -29,000 \times (95 - 100) = +145,000 \\ \mathit{Total} &= +50,000 \end{aligned}$$

5. If you wanted to rehedge, with the new delta, you should only be short

$$100,000 \times 0.10 = 10,000.$$

You have to buy back 19,000 of the shares you shorted.

A2: Gamma Scalping — The Intuition

You are **long a call** and **delta-hedged** (short Δ shares). Watch what happens as the stock wiggles:

1. Stock **rises**. Your call's delta rises (gamma > 0), so you are now *under-hedged*. You sell more shares — **at the higher price**.
2. Stock **falls back**. Your delta falls, so you are now *over-hedged*. You buy shares back — **at the lower price**.
3. Net of the round trip: you **sold high and bought low**. A small profit.

This is "gamma scalping." Positive gamma turns volatility into trading profit — the more the stock wiggles, the more you scalp.

But it's not free: every day you hold the option, you pay **theta** (time decay). Gamma giveth, theta taketh away.

A2: Gamma Scalping — A Two-Move Numerical Example

Long 1 call, $S_0 = 100$, $\Delta = 0.50$, $\Gamma = 0.04$. Delta-hedge: short 0.50 shares.

Move 1: stock rises to $102.

- New delta $\approx 0.50 + 0.04(2) = 0.58$. You're short only 0.50 $\Rightarrow$ sell 0.08 more shares at $102.

Move 2: stock falls back to $100.

- New delta $\approx 0.58 - 0.04(2) = 0.50$. You're short 0.58 $\Rightarrow$ buy back 0.08 shares at $100.

Scalp profit from the round trip: sold 0.08 sh @102, bought 0.08 sh @100:

$$0.08 \times (102 - 100) = \boxed{\$0.16}.$$

The bigger the swings, the more you scalp. Over the option's life you must out-scalp the theta you pay.

A2: Gamma Scalping — Realized vs Implied Vol

To get the option's whole-life P&L, we sum the scalp over *every* rebalancing step.
Per rebalancing step $[t,\ t+\Delta t]$ with price move ΔS:

$$\Delta\Pi \;\approx\; \underbrace{\tfrac{1}{2}\Gamma\,(\Delta S)^2}_{\text{gamma scalp (gain)}} \;-\; \underbrace{|\Theta|\,\Delta t}_{\text{time decay (cost)}}.$$

Summed over all steps to expiry:

$$\text{Total P\&L} \;\approx\; \underbrace{\tfrac{1}{2}\Gamma\sum_i(\Delta S_i)^2}_{\text{realized quadratic variation}} \;-\; \underbrace{|\Theta|\,T}_{\text{total decay paid}}.$$

- $\sum(\Delta S_i)^2$ is driven by how much the stock *actually* moves — the **realized volatility** σ_{real}.
- $|\Theta|\ T$ was fixed when you bought the option — it reflects the **implied volatility** σ_{imp} you paid.
- Long-gamma P&L $> 0 \iff \sigma_{\text{real}} > \sigma_{\text{imp}}$.

Buying an option and delta-hedging is a **bet that the stock will move more than the market priced in**. This is what "trading volatility" means.

A2: The Θ–Γ Identity — Why Gamma Is Never Free

Recall the Black-Scholes PDE (Lec 10), written with Greeks:

$$\Theta + rS\Delta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = rV.$$

For a **delta-neutral** portfolio ($\Delta = 0$):

$$\boxed{\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = rV}$$

- Time decay Θ and curvature Γ are **mechanically linked**.
- Long gamma ($\Gamma > 0$) forces $\Theta < 0$: you pay time decay to own convexity.
- Short gamma ($\Gamma < 0$) gives $\Theta > 0$: you earn time decay but carry curvature risk
- "There is no free lunch": the value of gamma scalping is exactly offset, on average, by the theta you pay — unless realized vol beats implied.

A3: Principal-Protected Note (PPN) — Structure

Remember A retail-marketed note that guarantees the principal back at maturity, with extra payout if a reference index has risen.

Generic decomposition into traded instruments:

$$\text{PPN} \;=\; \text{ZCB (par at } T) \;+\; \alpha \cdot \text{call(s) on the index.}$$

- **Buyer:** risk-averse retail / pension. Wants upside without downside.
- **Seller:** the bank (a market maker, "MM"). *Short* the structure $\Rightarrow$ must hedge.

Korean **ELB** (Equity-Linked Bond), linked to KOSPI 200 or single names, has exactly this shape

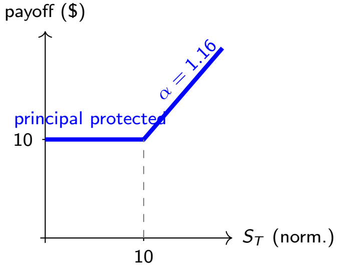
A3: Example — S&P 500 Note, Issued Feb 2008

Term sheet. You (the MM) **sell** a 7-year capital-protected note.

- Issue date 22 Feb 2008; maturity 20 Feb 2015.
- Issue price = principal = \$10. Coupon: 0%. Principal protection: 100%.
- Maturity payoff $= \$10 + \mathrm{SRA}$, where

$$\mathrm{SRA} \;=\; \$10 \times \underbrace{116\%}_{\text{participation } \alpha} \times \frac{\mathsf{Index}_T - \mathsf{Index}_0}{\mathsf{Index}_0}.$$

- Index: S&P 500, normalized so $S_0 = \$10$ ($S\&P = 1353.1$ on 28 Feb 08 $\Rightarrow \beta = 10/1353.1$).



The kink at $S_T = \$10$ is the embedded call's strike.

- **Below \$10:** the bond floors the payoff — the investor recovers full principal.
- **Above \$10:** the embedded calls capture $\alpha = 116\%$ of the index gain.

A3: BSM Valuation of the Note

Q. On 28 Feb 2008, with $r = 3.23\%$ c.c., $q = 2\%$, $\sigma = 15\%$, what is the fair value of the note? Issued at \$10 — is the bank making or losing money?

A. Decompose into ZCB + 1.16 ATM calls on the normalized index ($K = S_0 = 10,\ T = 7$).

Bond piece.

$$B_0 = 10 \cdot e^{-rT} = 10 \cdot e^{-0.2261} = \boxed{\$7.98}.$$

Call piece. $d_1 = 0.4154$, $d_2 = 0.0185$; $N(d_1) = 0.6611$, $N(d_2) = 0.5074$.

$$C_0 = 10e^{-qT}N(d_1) - 10e^{-rT}N(d_2) = 5.747 - 4.047 = \boxed{\$1.70}.$$

Note value.

$$V_0 = B_0 + \alpha \cdot C_0 = 7.98 + 1.16 \cdot 1.70 = \boxed{\$9.95}.$$

Issuer's margin. Sells at \$10, fair value \$9.95 $\Rightarrow$ gross fee $\approx$ \$0.05.

A3: Hedging the Short Note — Replicating Portfolio

Q. You sold the note (collected \$10). How do you hedge?

A. Build a **long** replicating portfolio with the same payoff. Two pieces:

1. Buy a **ZCB** for \$7.98 to hedge the principal at T.
2. Buy **1.16 synthetic calls**, each replicated as: long Δ shares + borrow.

Per call, today. The call's delta is $\Delta = e^{-qT} N(d_1) = \boxed{0.5747}$.

- Long the index: $\Delta \cdot S_0 = 0.5747 \cdot \$10 = \boxed{\$5.747}$.
- Borrow to fund the gap: $C_0 - \Delta S_0 = 1.70 - 5.747 = \boxed{-\$4.047}$ (i.e., short bond \$4.047).
- Net value $= 5.747 - 4.047 = \$1.70$. ✓

Scale by 1.16 to match the note: hold $1.16 \cdot 0.5747 = 0.667$ in the index per dollar of principal, plus the protective ZCB.

A3: Dynamic Delta Hedging — Keeping the Hedge Alive

The replicating portfolio is **not static**: Δ moves with S, with t, and with any change in r or σ.

Daily routine of the issuer's desk.

1. Recompute $\Delta = e^{-qT}\,N(d_1)$ at today's S_t and remaining time $T - t$.
2. Trade the underlying (or index futures) to match the new Δ.
3. Adjust the bond/financing leg so the portfolio still matches the option value.

Effectiveness depends on:

- Frequency of rebalancing (continuous in theory; daily in practice).
- Stability of σ — a vol spike means today's Δ was wrong.
- No jumps — a gap move can't be hedged after the fact.
- Transaction costs and liquidity — KOSPI 200 futures are deep, but single-stock ELB hedges face wider spreads.

A4: Portfolio Insurance — The LOR Story

1981. UC Berkeley professors **Hayne Leland** and **Mark Rubinstein** partnered with **John O'Brien** to found **LOR Associates**.

The idea. Use option-pricing theory to *synthesize a protective put* on an equity portfolio — without ever buying a listed put.

- Sell $-\Delta_{\text{put}}\%$ of equity, invest the rest in risk-free assets; rebalance daily as Δ moves with the market.
- Market falls $\Rightarrow$ Δ_{put} becomes more negative $\Rightarrow$ *sell stock, buy cash*.
- Market rises $\Rightarrow$ *buy stock back*.

Adoption. Marketed to pension funds, endowments, mutual funds. Slow 1981–83, then explosive: by 1987, $\sim$\$100**B** of US equity assets ran portfolio insurance.

Why this matters here. Portfolio insurance is the original real-world delta-hedging program at institutional scale — and the original cautionary tale about what happens when the machinery meets a jump.

A4: Example — Synthesizing a Protective Put

Q. You manage a \$90M equity portfolio and want to floor its 6-month value at \$87M. Build the synthetic put.
Inputs: $S_0 = 90$, $K = 87$, $r = 9\%$, $q = 3\%$, $\sigma = 25\%$, $T = 0.5$.

A.

$$d_1 = \frac{\ln(90/87) + (0.09 - 0.03 + 0.03125)(0.5)}{0.25\sqrt{0.5}} = \frac{0.0795}{0.1768} = \boxed{0.4499}.$$

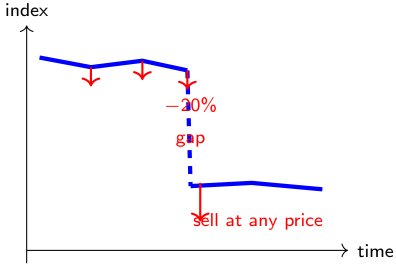
$$\Delta_{\text{put}} = e^{-qT}\big(N(d_1) - 1\big) = e^{-0.015}(0.6736 - 1) = \boxed{-0.3215}.$$

Initial hedge: Sell $-\Delta_{\text{put}} = 32.15\%$ of equity ($\approx$\$28.9M) into risk-free assets.

Next day, market falls to \$88M. Recomputing: $\Delta_{\text{put}} \approx -0.3679$.
$\boxed{\text{sell another } 4.64\%}$ ($\approx \$4M$) into risk-free assets.

Note the pattern. Market *falls* ⇒ desk *sells more* (negative gamma of the synthetic put). On a smooth path this works. On a gap, it doesn't.

A4: When Hedging Fails — Black Monday, 19 Oct 1987



Red arrows: forced selling triggered by negative gamma.

- The S&P 500 fell $\boxed{-20.5\%}$ *in one day*.
- Portfolio-insurance programs at ∼\$100B AUM demanded continuous selling as the index dropped.
- Markets gapped — there was no continuous path to hedge along.
- **Feedback loop:** forced sell orders pushed prices lower ⇒ delta required more selling ⇒ more downward pressure.
- LOR Associates entered receivership by 1988.
- The **Brady Commission** cited dynamic-hedging programs as a major amplifier of the crash.

The lesson. Delta-hedging is risk management *conditional* on continuous, liquid markets. When that assumption breaks, the hedge *is* the risk.

A5: Delta-Gamma Hedging — Why

Limits of pure delta hedging.

- Delta is a first-order tangent. As soon as S moves, Δ drifts by $\Gamma\,\Delta S$ — so the hedge degrades with the size of the move.
- Bigger moves $\Rightarrow$ either more frequent rebalancing (high transaction cost) or larger tracking errors.
- A *jump* kills the hedge entirely.

Fix: add a second instrument that carries gamma.

- Stock has *zero* gamma $\Rightarrow$ no amount of stock can offset gamma. **You need an option to hedge an option's curvature.**
- Pick a liquid, short-dated traded option C_1 (high Γ_1) and use it to zero out the portfolio gamma.
- Result: a portfolio that is locally *flat* in S to second order — robust over a much wider price range before rebalancing is needed.

A5: The System and Its Solution

Portfolio (We have short the long-dated call C. Create a hedged portfolio):

$$\Pi = -C(S,T) + N \cdot S + N^C \cdot C_1(S,T_1).$$

Two hedging conditions:

$$\begin{aligned} \partial\Pi/\partial S &= -\Delta + N + N^C\Delta_1 = 0 \qquad &\text{(delta-neutral)} \\ \partial^2\Pi/\partial S^2 &= -\Gamma + N^C\Gamma_1 = 0 \qquad &\text{(gamma-neutral)} \end{aligned}$$

Solve in order:

$$\boxed{N^C = \frac{\Gamma}{\Gamma_1}, \qquad N = \Delta - N^C\Delta_1}$$

Why this works.

- Take a *long* position in C_1 ($N^C > 0$): it carries positive gamma to offset the short C.
- The stock position N is *smaller* than in delta-only hedging because the long C_1 already contributes some delta ($N^C\Delta_1$).

A5: Example — PPN Hedge Continued

Q. You are short the 7-year ATM call embedded in the PPN ($S_0 = 10$, $K = 10$, $r = 3.23\%$, $q = 2\%$, $\sigma = 15\%$). Hedge it with a 1-year ATM traded call.

Given Greeks:

	Price	Δ	Γ
Long-dated call C (the liability)	1.7000	0.5747	0.0801
1-year traded call C_1	0.6443	0.5512	0.2575

A. Solve:

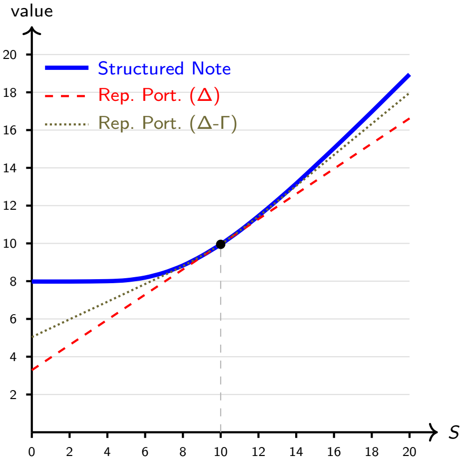
$$N^C = \frac{\Gamma}{\Gamma_1} = \frac{0.0801}{0.2575} = \boxed{0.3113} \quad \text{(long } C_1\text{)},$$

$$N = \Delta - N^C \Delta_1 = 0.5747 - 0.3113 \cdot 0.5512 = \boxed{0.4031} \quad \text{(long stock)},$$

$$B = C - NS - N^C C_1 = 1.70 - 4.031 - 0.2006 = \boxed{-\$2.53} \quad \text{(borrow)}.$$

Compare to delta-only. Pure delta: long 0.5747 shares. Delta-gamma: long 0.4031 shares + 0.3113 short-dated calls — the option carries part of the directional exposure, so we hold *less* stock.

A5: Why Gamma-Hedging Helps — and Where It Still Fails



The picture. Blue curve is the true option value $V(S)$. Red dashed line is the delta tangent at S_0. Green dotted parabola adds the gamma term and matches V much more closely over a wider band.

Practical.

- Less frequent share rebalancing $\Rightarrow$ lower transaction costs on the stock leg.
- But the C_1 option leg *also* has to be rebalanced — use liquid, exchange-traded options.
- Survives larger swings; *still* dies on a true jump.
- Does *not* hedge vega — a vol spike still hurts (next: vega-gamma).

A6: Vega Hedging — Same Machinery, Different Greek

Vega measures sensitivity to a parallel shift in implied volatility: $\nu = \partial V / \partial \sigma$.
Stock has *zero* vega — so, as with gamma, we need a *traded option* to hedge it.

Same portfolio as before:

$$\Pi = -C(S, T) + N \cdot S + N^C \cdot C_1(S, T_1).$$

Vega-neutral condition:

$$\frac{\partial \Pi}{\partial \sigma} = -\nu + N^C \nu_1 = 0 \quad \Rightarrow \quad \boxed{N^C = \frac{\nu}{\nu_1}}$$

- Long-dated options carry *more vega* than short-dated ones ($\nu \propto S\sqrt{T}$) $\Rightarrow$ typically $N^C > 1$ when C_1 is short-dated.
- **Problem:** this N^C generally does *not* also zero out gamma. Setting $N^C = \nu/\nu_1$ and $N^C = \Gamma/\Gamma_1$ at the same time requires $\nu_1/\nu = \Gamma_1/\Gamma$ — not generally true.
- **One traded option can hedge gamma or vega, not both**. To hedge both, we need two options (next slide).

A6: Vega-Gamma Hedging — Two Options

Add a second traded option C_2 with a different (Γ_2, ν_2) profile (usually a different maturity or strike):

$$\Pi = -C + N\,S + N_1^C\,C_1 + N_2^C\,C_2.$$

Two conditions for two unknowns:

$$\begin{aligned} \text{Gamma-neutral:} \quad & -\Gamma + N_1^C\Gamma_1 + N_2^C\Gamma_2 = 0, \\ \text{Vega-neutral:} \quad & -\nu + N_1^C\nu_1 + N_2^C\nu_2 = 0. \end{aligned}$$

Solve the 2×2 system:

$$N_2^C = \frac{\nu\,\Gamma_1 - \nu_1\,\Gamma}{\nu_2\,\Gamma_1 - \nu_1\,\Gamma_2}, \quad N_1^C = \frac{\Gamma - N_2^C\Gamma_2}{\Gamma_1}.$$

Then set delta with the stock leg: $N = \Delta - N_1^C\Delta_1 - N_2^C\Delta_2$.

Why this works. Two options with *different* Γ/ν ratios span the 2-d (Γ, ν) plane. The stock has zero in both coordinates — it can only hedge delta, the leftover dimension.

Need more Greeks neutralized (rho, vanna, volga)? Add more options. General rule: **one independent instrument per risk dimension**.

A6: Example — Gamma + Vega Neutral

Q. Your portfolio is already delta-neutral but has $\Gamma = -5{,}000$ and $\nu = -8{,}000$ (short-vol exposure, e.g. from selling options). Two traded options are available:

	Δ	Γ	ν
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

Find quantities w_1, w_2 to make the portfolio gamma- *and* vega-neutral, and the share trade to restore delta-neutrality.

A. The two conditions:

$$-5{,}000 + 0.5w_1 + 0.8w_2 = 0$$
$$-8{,}000 + 2.0w_1 + 1.2w_2 = 0$$

Solve $\Rightarrow$ $\boxed{w_1 = 400,\ w_2 = 6{,}000}$.

Residual delta from the options: $400(0.6) + 6{,}000(0.5) = 240 + 3{,}000 = 3{,}240$.
$\Rightarrow$ $\boxed{\text{Sell 3,240 units of the underlying}}$ to restore $\Delta = 0$.

The stock trade does *not* affect gamma or vega (both are zero for stock).

General Approach — Hedging as Linear Algebra

Two facts make Greek-based hedging tractable.

1. **Greeks add linearly across a book:** $\Delta_\Pi = \sum_i \Delta_i$, same for $\Gamma, \nu, \rho, \ldots$
2. **Hedging is solving a linear system:** n independent risk dimensions $\Rightarrow$ need $\geq n$ instruments with *linearly independent* Greek profiles.

Which instrument carries which Greek?

Instrument	Δ	Γ	ν	ρ	Can hedge ...
Cash / bond	0	0	0	yes	rates only
Stock / index	1	0	0	0	delta only
Forward	≈ 1	0	0	small	delta only
Vanilla option (any K, T)	yes	yes	yes	yes	all four
Two options, *same* K, T	identical profiles				one dimension only

Rule of thumb. Want to hedge n Greeks? Pick n instruments with visibly *different* maturities or strikes.

When the Hedge Is Non-Unique — Choose Wisely

If you have *more* instruments than risks, the system is under-determined: a whole **family** of weights satisfies $\Delta = \Gamma = \nu = \cdots = 0$.

Use the leftover degrees of freedom to optimize a secondary objective.

Objective	How the trader uses it
Minimize hedge cost	Cheapest mix of options + stock that hits the constraints.
Capture mispricing	Over- (under-)weight options trading rich (cheap)
Minimize future rebalancing	Pick weights whose Greeks drift slowly with S, t, σ.
Minimize market impact	Use the most liquid options; avoid thin strikes/maturities.
Minimize counterparty risk	Favor exchange-traded over OTC where possible.

Practitioners' choice = the hedge that best balances all these. Often a quadratic program: minimize transaction cost subject to Greek constraints.

Appendix: More Applications

Delta-Hedging with Forwards and Futures

Forwards and futures both have delta close to 1 — but not identical.

Instrument	Value f_t	$\Delta = \partial f_t / \partial S_t$	Capital required
Stock	S_t	1	full price
Forward (OTC)	$S_t - Ke^{-r(T-t)}$	1	zero
Futures (exchange)	$S_t\, e^{(r-q)(T-t)}$	$e^{(r-q)(T-t)} > 1$	margin only

Conversion rule. To replicate a spot delta of Δ:

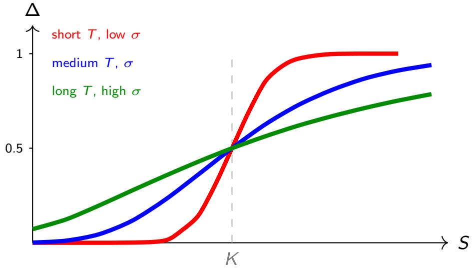
- Use Δ *forwards*, OR
- Use $e^{-(r-q)(T-t)} \cdot \Delta$ *futures* (fewer than forwards because each future carries more than 1 unit of spot exposure).

Example. A US bank's FX option book needs a delta hedge equivalent to short £458,000 spot. $r_{\text{US}} = 4\%$, $r_{\text{UK}} = 7\%$, 9-month GBP futures.

$$N_{\text{fut}} = e^{-(0.04-0.07)(0.75)} \times 458{,}000 = e^{0.0225} \times 458{,}000 = 468{,}442.$$

At £62,500 per contract: $468{,}442/62{,}500 = 7.495 \Rightarrow$ short $\boxed{\text{7 contracts}}$

Delta's Shape — How T and σ Bend the Curve



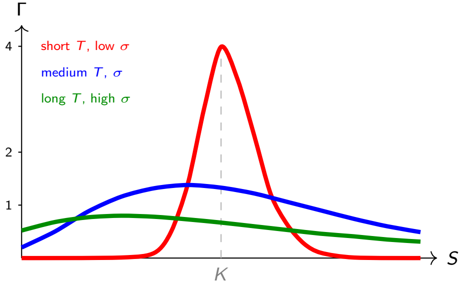
The shape of $\Delta(S)$ is governed by $\sigma\sqrt{T}$.

- **Small** $\sigma\sqrt{T}$ (short maturity *or* low vol): Δ approaches a digital step at K. The option "picks a direction" early.
- **Large** $\sigma\sqrt{T}$ (long maturity *or* high vol): Δ is gentle and spread out. Even far-OTM options have non-trivial delta because they could still finish ITM.
- All three pass through $\Delta = 0.5$ at $S = K$ (approximately).

Intuition. Higher vol $\Rightarrow$ all strikes are "reachable" $\Rightarrow$ probabilities compress toward 0.5 $\Rightarrow$ delta flattens.

Hedging implication. Short-dated ATM options have the most volatile delta — the rebalance frequency must rise as expiry approaches. This is the "Δ-cliff" phenomenon on options expiry day.

Gamma's Shape — The Bell Narrows as $T \to 0$



Peak gamma scales as $1/(\sigma\sqrt{T})$.

$$\Gamma_{\text{ATM}} \approx \frac{1}{S\sigma\sqrt{T}\,\sqrt{2\pi}}.$$

Shape rules:

- **Short T / low σ:** tall, narrow spike centered at K. ATM gamma can be enormous as expiry nears.
- **Long T / high σ:** low, wide bell. Gamma is spread across a broad range of strikes; no single strike dominates.
- Total "area" under the gamma curve is roughly conserved as T or σ change — spreading out comes at the cost of height.

Why this matters. Short-dated ATM gamma is what *kills* delta hedgers on expiry days: a small move in S implies a large change in Δ, demanding frantic rebalancing. Mid-month rolls (long T) are gentle; expiry weeks (short T) are not.