BUSS386 Problem Set 10 — Solutions

Black-Scholes-Merton

Prof. Ji-Woong Chung

Problem 1 — Vanilla call on KOSPI 200

$S_0 = 360, K = 380, T = 0.25, r = 0.03, \sigma = 0.18$, no dividend.

(a) $\ln(360/380) = -0.05407$. $\sigma\sqrt{T} = 0.18 \cdot 0.5 = 0.09$.
$d_1 = (-0.05407 + (0.03 + 0.0162) \cdot 0.25)/0.09 = (-0.05407 + 0.01155)/0.09 = \boxed{-0.4725}$.
$d_2 = -0.4725 - 0.09 = \boxed{-0.5625}$.

(b) $N(-0.4725) = 0.3183$, $N(-0.5625) = 0.2869$.
$c = 360 \cdot 0.3183 - 380 \cdot e^{-0.0075} \cdot 0.2869 = 114.59 - 108.21 = \boxed{6.38}$ pts.
Per contract: $6.38 \cdot 250{,}000 = \boxed{₩1{,}595{,}000}$.

(c) **BSM put:** $N(0.4725) = 0.6817$, $N(0.5625) = 0.7131$.
$p = 380e^{-0.0075} \cdot 0.7131 - 360 \cdot 0.6817 = 268.96 - 245.41 = \boxed{23.55}$ pts.
Parity: $p = c + Ke^{-rT} - S_0 = 6.38 + 377.16 - 360 = 23.54$ pts. Matches within rounding. ✓

(d) $\mathbb{Q}(\text{exercise}) = N(d_2) = \boxed{28.7\%}$. Delta $= N(d_1) = \boxed{0.318}$.

Problem 2 — Making sense of option prices in the market

$S_0 = 147.80$, $K = 130$, $T = 5/12 = 0.4167$, $r = 0.005$ c.c., $\sigma = 0.30$, $n = 10$ steps so $\Delta t = T/n = 0.04167$.

(a) $u = e^{\sigma\sqrt{\Delta t}} = e^{0.30 \cdot 0.2041} = e^{0.06124} = \boxed{1.0632}$.
$d = 1/u = \boxed{0.9406}$.
$e^{r\Delta t} = e^{0.000208} = 1.000208$.
$p = (e^{r\Delta t} - d)/(u - d) = (1.000208 - 0.9406)/0.1226 = \boxed{0.4861}$.

(b) Using $u \cdot d = 1$, $S_T(j) = S_0 \cdot u^{2j-10}$ and $\Pr(j) = \binom{10}{j} p^j (1-p)^{10-j}$:

j	$S_T(j)$	Payoff $\max(S_T - 130, 0)$	$\binom{10}{j} p^j (1-p)^{10-j}$
0	80.02	0	0.0013
1	90.53	0	0.0122
2	102.34	0	0.0518
3	115.67	0	0.1306
4	130.76	0.76	0.2162
5	147.80	17.80	0.2453
6	167.07	37.07	0.1932
7	188.85	58.85	0.1044
8	213.49	83.49	0.0370
9	241.33	111.33	0.0078
10	273.00	143.00	0.0007

(c) $E^{\mathbb{Q}}[\text{payoff}] = \sum_j \Pr(j)\,\text{payoff}_j = 0.2162(0.76) + 0.2453(17.80) + 0.1932(37.07) + 0.1044(58.85)$ $+0.0370(83.49) + 0.0078(111.33) + 0.0007(143.00) = 0.164 + 4.366 + 7.162 + 6.144 + 3.089 + 0.866 + 0.105 = 21.90$.
Discount: $e^{-rT} = e^{-0.002083} = 0.99792$.
Price $= 0.99792 \cdot 21.90 = \boxed{\$21.85}$.

(d) Market price is \$20.00; 10-step binomial estimate is \$21.85 — the model *over-prices* the call by about \$1.85 (9%). The most likely fix: the assumed historical $\sigma = 30\%$ is too high for the period in question; backing σ out from the market price (implied volatility) and re-pricing the other strikes would close the gap.

(e) BSM directly with the same inputs: $\sigma\sqrt{T} = 0.30 \cdot 0.6455 = 0.1937$.
$d_1 = (\ln(147.80/130) + (0.005 + 0.045) \cdot 0.4167)/0.1937 = (0.1284 + 0.0208)/0.1937 = \boxed{0.7706}$.
$d_2 = 0.7706 - 0.1937 = \boxed{0.5769}$.
$N(d_1) = 0.7796,\ N(d_2) = 0.7180$.
$c = 147.80 \cdot 0.7796 - 130 \cdot 0.99792 \cdot 0.7180 = 115.23 - 93.20 = \boxed{\$22.03}$.
The 10-step binomial (\$21.85) lies a few cents below BSM (\$22.03) — expected, since binomial $\to$ BSM as $N \to \infty$.

(f) Try lowering σ:

σ	BSM call price
30.0%	22.03
25.0%	20.66
22.0%	19.96
20.5%	19.65

Implied vol $\approx \boxed{22\%}$ (BSM at 22% gives \$19.96, essentially the market \$20.00).
Lesson: historical σ of 30% over-states what the market is pricing. The market's *implied* vol on this strike is closer to 22%. The full IV surface and its variation across strikes (the smile) is Lec 12.

Problem 3 — Merton (continuous dividend)

$S_0 = 360, K = 360, T = 0.5, r = 0.03, \sigma = 0.14, q = 0.015$.

(a) $F_0 = 360e^{(0.03-0.015)\cdot 0.5} = 360 \cdot e^{0.0075} = 360 \cdot 1.00753 = \boxed{362.71}$.

(b) $\sigma\sqrt{T} = 0.14 \cdot \sqrt{0.5} = 0.09899$.
$d_1 = (\ln(360/360) + (0.03 - 0.015 + 0.0098)(0.5))/0.09899$
$= (0 + 0.0248 \cdot 0.5)/0.09899 = 0.0124/0.09899 = \boxed{0.1253}$.
$d_2 = 0.1253 - 0.0990 = \boxed{0.0263}$.

(c) $N(d_1) = N(0.1253) = 0.5499,\ N(d_2) = N(0.0263) = 0.5105$.
$c = 360e^{-0.0075} \cdot 0.5499 - 360e^{-0.015} \cdot 0.5105 = 196.48 - 181.04 = \boxed{15.44}$ pts.

(d) With $q = 0$: $c = 16.95$ pts (lecture worked example). With $q = 1.5\%$: $c = 15.44$.
Difference $= \boxed{-1.51}$ pts. Dividends transfer value from the stock to the dividend stream, which the call holder doesn't receive — so the effective "stock" for the option becomes S_0e^{-qT}, reducing the call's value.

Problem 4 — USD/KRW currency call

$S_0 = 1380, K = 1400, T = 0.5, r = 0.025, r_f = 0.040, \sigma = 0.10.$

(a) $F_0 = 1380 \cdot e^{(0.025-0.040)(0.5)} = 1380 \cdot e^{-0.0075} = 1380 \cdot 0.99253 = \boxed{₩\,1{,}369.7}$.

(b) $\sigma\sqrt{T} = 0.07071$. $\ln(1380/1400) = -0.01439$.
$d_1 = (-0.01439 + (0.025 - 0.040 + 0.005)(0.5))/0.07071 = (-0.01439 - 0.005)/0.07071 = \boxed{-0.2742}$.
$d_2 = -0.2742 - 0.0707 = \boxed{-0.3449}$.
$N(-0.2742) = 0.3920,\ N(-0.3449) = 0.3651$.
$c = 1380 \cdot 0.9802 \cdot 0.3920 - 1400 \cdot 0.9876 \cdot 0.3651 = 530.25 - 504.81 = \boxed{₩\,25.4}$ per USD.

(c) $N(0.2742) = 0.6080,\ N(0.3449) = 0.6349$.
$p = 1400 \cdot 0.9876 \cdot 0.6349 - 1380 \cdot 0.9802 \cdot 0.6080 = 877.81 - 822.43 = \boxed{₩\,55.4}$ per USD.
Parity: $p + S_0 e^{-r_f T} = 55.4 + 1352.7 = 1408.1;\quad c + Ke^{-rT} = 25.4 + 1382.6 = 1408.0$. Matches ✓.

(d) $r_{\text{USD}} > r_{\text{KRW}}$, so USD has higher carry. By covered interest parity, USD trades at a forward **discount** ($F_0 < S_0$), which is what we see (1369.7 ¡ 1380).

Problem 5 — Black '76 on KTB futures

$F_0 = 102.50, K = 102.00, T = 0.25, r = 0.03, \sigma = 0.06.$

(a) $c = e^{-rT}\left[F_0 N(d_1) - K N(d_2)\right]$ with $d_1 = (\ln(F_0/K) + \sigma^2 T/2)/(\sigma\sqrt{T})$, $d_2 = d_1 - \sigma\sqrt{T}$.

(b) $\ln(102.5/102) = 0.004890$. $\sigma\sqrt{T} = 0.06 \cdot 0.5 = 0.030$. $\sigma^2 T/2 = 0.00045$.
$d_1 = (0.004890 + 0.000450)/0.030 = \boxed{0.1780}$.
$d_2 = 0.1780 - 0.030 = \boxed{0.1480}$.
$N(d_1) = 0.5707,\ N(d_2) = 0.5588$.
$c = e^{-0.0075}[102.50 \cdot 0.5707 - 102.00 \cdot 0.5588] = 0.99253 \cdot (58.50 - 56.99) = 0.99253 \cdot 1.51 = \boxed{1.50}$ per ₩ 100M face.

(c) Setting $q = r$ in the Merton formula is exactly what produces Black '76: in the risk-neutral world a futures has zero drift ($E^{\mathbb{Q}}[F_T] = F_0$), the same as a stock with continuous dividend yield equal to r (whose risk-neutral drift is $r - q = 0$).

Problem 6 — Verifying the BSM PDE

PDE: $V_t + rSV_S + \frac{1}{2}\sigma^2 S^2 V_{SS} = rV$.

(a) $V = S^2 e^{(r+\sigma^2)(T-t)}$. Terminal at $t = T$: $V = S^2$. ✓
$V_t = -(r + \sigma^2)V,\ V_S = 2V/S,\ V_{SS} = 2V/S^2$.
PDE LHS: $-(r + \sigma^2)V + rS(2V/S) + \frac{1}{2}\sigma^2 S^2 (2V/S^2) = -(r + \sigma^2)V + 2rV + \sigma^2 V = rV$. $\boxed{\checkmark}$

(b) $V = e^{-r(T-t)}$. Terminal: $V(T) = 1$. ✓
$V_t = re^{-r(T-t)} = rV,\ V_S = V_{SS} = 0$.
PDE LHS: $rV + 0 + 0 = rV$. $\boxed{\checkmark}$

(c) Ansatz $V(S,t) = S^n \cdot h(T-t)$ with $h(0) = 1$ (terminal condition).
$V_t = -S^n h'(T-t) = -h'/h \cdot V$. $V_S = nS^{n-1}h = nV/S$. $V_{SS} = n(n-1)V/S^2$.
PDE: $-h'/h \cdot V + rnV + \frac{1}{2}\sigma^2 n(n-1)V = rV$
$\Rightarrow h'/h = rn + \frac{1}{2}\sigma^2 n(n-1) - r = r(n-1) + \frac{1}{2}\sigma^2 n(n-1)$.
With $\tau = T - t$ so $dh/d\tau = -h'$: $h(\tau) = e^{[r(n-1)+\frac{1}{2}\sigma^2 n(n-1)]\tau}$.
$\boxed{V(S,t) = S^n \exp\big[(r(n-1) + \frac{1}{2}\sigma^2 n(n-1))(T-t)\big]}$.
Sanity check: $n = 1 \Rightarrow$ coefficient $= 0$, so $V = S$ (the stock). $n = 2 \Rightarrow$ coefficient $= r + \sigma^2$, recovering part (a). ✓

Problem 7 — Cash-or-nothing digital

(a) Payoff $= \mathbb{1}\{S_T \geq K\}$. By risk-neutral pricing,

$$V_0 = e^{-rT} E^{\mathbb{Q}}[\mathbb{1}\{S_T \geq K\}] = e^{-rT}\mathbb{Q}(S_T \geq K).$$

(b) From the lecture (or Problem 1): $\mathbb{Q}(S_T \geq K) = N(d_2)$. So $\boxed{V_0^{\text{digital call}} = e^{-rT}N(d_2)}$.

(c) $\sigma\sqrt{T} = 0.25$. $d_1 = (0 + (0.05 + 0.03125)(1))/0.25 = 0.325$. $d_2 = 0.325 - 0.25 = 0.075$.
$N(0.075) = 0.5299$.
$V_0 = e^{-0.05} \cdot 0.5299 = 0.9512 \cdot 0.5299 = \boxed{\$0.504}$.

(d) Vanilla call price (same parameters): $c = 100N(0.325) - 100e^{-0.05}N(0.075) = 100 \cdot 0.6274 - 95.12 \cdot 0.5299 = 62.74 - 50.41 = \$12.33$. Digital pays a fixed \$1 in the up state; vanilla pays $S_T - K$ which is unbounded above. So vanilla is much larger. The digital is essentially $e^{-rT}N(d_2)$ — a discounted probability, capped at $e^{-rT} \cdot 1$.