BUSS386 Problem Set 10

Black-Scholes-Merton

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Problem 1 — Vanilla call on KOSPI 200

$S_0 = 360$, $K = 380$, $T = 0.25$, $r = 3\%$ c.c., $\sigma = 18\%$, no dividend. KRX multiplier ₩250,000 per index point.

(a) Compute d_1 and d_2.

(b) Compute the call price per index point and per contract (in ₩).

(c) Compute the put price two ways: (i) the BSM put formula directly, (ii) put-call parity. Confirm they match.

(d) State the risk-neutral probability of exercise of the call. State the call's delta.

Problem 2 — Making sense of option prices in the market

On 12 November 2020, the market prices of call options on Johnson & Johnson stock are as follows. The expiration date of these options is 16 April 2021.

Strike price	Market price	Strike price	Market price
130	20.00	150	5.50
135	15.60	155	3.50
140	11.60	160	2.10
145	8.20	165	1.20

The stock price on 12 November 2020 was \$147.80. The risk-free rate is $r = 0.5\%$ per annum (c.c.). Time to expiration is $T = 5/12$ year. Suppose volatility is estimated to be $\sigma = 30\%$ per annum, and a 10-step binomial tree is used to estimate the price of the $K = 130$ call.

(a) Calculate u, d, and the risk-neutral probability p.

(b) Construct a table showing, for each terminal node $j = 0, 1, \ldots, 10$: the stock price $S_T(j)$, the call payoff, and the risk-neutral probability.

(c) Compute the option price.

(d) Compare the theoretical price to the market price. How well does it match? In one sentence, what could we do to better match the market?

(e) Compute the same call price using the Black-Scholes formula directly with the same $\sigma = 30\%$. Compare to the 10-step binomial answer.

(f) By trial, find the σ that makes the BSM price equal the market price of \$20.00 (to the nearest 0.5%). This is the option's *implied volatility* — we'll formalize it in Lec 12.

Problem 3 — Merton (continuous dividend)

A 6-month European call on KOSPI 200, $S_0 = 360$, $K = 360$, $r = 3\%$ c.c., $\sigma = 14\%$, continuous dividend yield $q = 1.5\%$.

(a) Compute the forward price F_0.

(b) Compute d_1, d_2 using either S_0 with dividend correction or F_0 directly.

(c) Compute the call price.

(d) Repeat the computation with $q = 0$. By how much does the dividend reduce the call price? In one sentence, why?

Problem 4 — USD/KRW currency call (Garman-Kohlhagen)

A Korean exporter wants to hedge USD receivables with a USD call. Setup: spot $S_0 = ₩\,1{,}380$ per USD, strike $K = ₩\,1{,}400$, $T = 0.5$, $r_{\text{KRW}} = 2.5\%$, $r_{\text{USD}} = 4.0\%$, $\sigma_{\text{FX}} = 10\%$.

(a) Compute the forward USD/KRW rate F_0.

(b) Compute d_1, d_2 and the USD call price (in ₩ per USD).

(c) Compute the USD put price at the same K via Garman-Kohlhagen, then verify the FX parity $p + S_0 e^{-r_f T} = c + K e^{-rT}$.

(d) In one sentence, why is $S_0 < F_0$ here (USD trades at forward premium or discount)?

Problem 5 — Black '76 (option on futures)

A 3-month European call on the December KTB futures. $F_0 = 102.50$ (per ₩ 100M face), $K = 102.00$, $r = 3\%$ c.c., $\sigma = 6\%$.

(a) State the Black '76 formula.

(b) Compute the call price.

(c) In one sentence, why does the Black '76 formula not need a separate "dividend yield" input even though it sets $q = r$ implicitly?

Problem 6 — Verifying the BSM PDE

The BSM PDE is

$$\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} = rV.$$

Verify by direct substitution that each of the following candidate solutions satisfies the PDE, given the stated terminal condition.

(a) $V(S,T) = S^2$, $V(S,t) = S^2 e^{(r+\sigma^2)(T-t)}$.

(b) $V(S,T) = 1$, $V(S,t) = e^{-r(T-t)}$.

(c) $V(S,T) = S^n$ for a constant n. Find $V(S,t)$ of the form $S^n \cdot h(T-t)$ and identify h. (Hint: S^n is a monomial; the ansatz is a deterministic time multiplier.)

Problem 7 — Cash-or-nothing digital

A European *digital call* pays \$1 if $S_T \geq K$ and zero otherwise. (S_T is a non-dividend stock, BSM assumptions.)

(a) Use the risk-neutral pricing principle to express the digital's price as $e^{-rT} \cdot \mathbb{Q}(S_T \geq K)$.

(b) Conclude that the digital call price is $e^{-rT} N(d_2)$.

(c) For $S_0 = 100$, $K = 100$, $T = 1$, $r = 5\%$, $\sigma = 25\%$, compute the digital's price.

(d) How does the digital price compare to the corresponding vanilla call? Is it always smaller? Explain in one sentence.