Black-Scholes-Merton Model

BUSS386. Futures and Options

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Lecture Outline

- Black-Scholes-Merton Model
    - Log-Normal Property of Stock Prices
    - Derivation
    - Iterpretation

BSM Model

Binomial Model vs. Black–Scholes–Merton (BSM) Model

- **Binomial model:** assumes the underlying asset price moves in discrete time-steps (up or down at each step).

- **BSM model:** built on continuous-time dynamics, modelling the asset price as evolving continuously.

- Although their approaches differ, they are closely connected: as the binomial time-steps shrink toward zero, the discrete model *converges* to the BSM formula for European-style options.

- When to use which:
  - Use the binomial model for flexibility (e.g., American options, early exercise, variable volatility).
  - Use the BSM model when assumptions (continuous trading, no early exercise, constant volatility) are reasonable and a closed-form solution is desired.

Black-Scholes-Merton Model

- The prices of European call and put options on non-dividend-paying stock are

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and $N(x)$ is the cumulative probability distribution function for a standard normal random variable.

BSM Model – Distribution of Future Stock Price

- A core assumption of the Black–Scholes–Merton model (BSM) is that the underlying stock price follows a log-normal distribution, i.e.

$$\ln S_T \sim N(m, s^2)$$

  where $N(m, s^2)$ denotes a normal distribution with mean m and variance s^2.[1]

- We can also derive this log-normal result via the discrete-time binomial model:
  - As the time-step size tends to zero and the number of steps tends to infinity, the binomial distribution of stock-price paths converges to the continuous GBM model and hence to log-normal terminal distribution.

- Next, we will prove the log-normality of S_T.

[1] Equivalently, S_T is log-normally distributed, which ensures $S_T > 0$ and aligns with modelling via Geometric Brownian Motion (GBM):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

and thus

$$\ln S_T = \ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right) T + \sigma W_T.$$

Log-Normal Property of Stock Prices – Setup

- Consider a binomial tree for the stock price with n steps each of length $\Delta t = \dfrac{T}{n}$.
- At each step the stock either moves up by factor u or down by factor d.
- If there are j upward moves and $n - j$ downward moves, then at expiry

$$S_T(j) = S_0\, u^j\, d^{\,n-j}.$$

(This sets the discrete-time framework from which we will pass to a continuous-time limit.)

Distribution of Terminal Moves

- What will happen if n becomes infinitely large? (This would be equivalent to making each step infinitesimally small).
- To see this, let's increase the number of steps n.

Distribution of j when $n = 10$ and $p = 0.5$

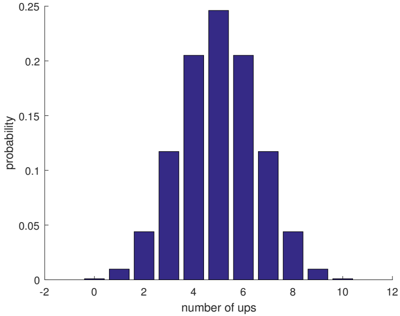
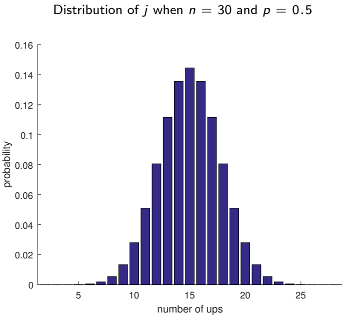
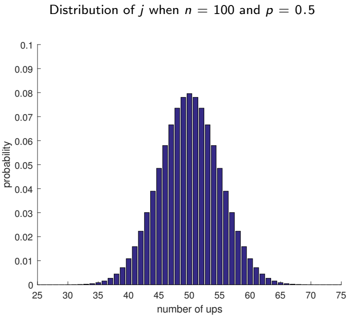


Illustration of Convergence





- As n grows large, the bar/histogram of j becomes more like a smooth bell curve. $\rightarrow$ The Central Limit Theorem

- Hence, as n approaches infinity, the number of upward movement will be normally distributed

$$j \sim N(np, \sqrt{np(1-p)}).$$

From Terminal Moves to Log Stock Price

- We now know the distribution of j. Next, let's find the distribution of $S_T(j)$.

- Using $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$,

$$\begin{aligned} S_T(j) &= S_0 u^j d^{n-j} \\ &= S_0 e^{(\sigma\sqrt{\Delta t})j} e^{(-\sigma\sqrt{\Delta t})(n-j)} \\ &= S_0 e^{(2\sigma\sqrt{\Delta t})j - n\sigma\sqrt{\Delta t}} \end{aligned}$$

- The log of stock price is

$$\ln S_T(j) = \ln S_0 + (2\sigma\sqrt{\Delta t})j - n\sigma\sqrt{\Delta t}$$

- As j is normally distributed, $\ln S_T$ is also normally distributed. Hence, $S_T(j)$ is log-normally distributed.

Mean and Variance of $\ln S_T$ (in limit)

- To further identify the distribution, let's find the mean and the standard deviation of $\ln S_T$.

- The mean of $\ln S_T$ is

$$\begin{aligned} E(\ln S_T) &= \ln S_0 + 2\sigma\sqrt{\Delta t}E(j) - n\sigma\sqrt{\Delta t} \\ &= \ln S_0 + 2\sigma\sqrt{\Delta t}(np) - n\sigma\sqrt{\Delta t} \end{aligned}$$

Mean and Variance of $\ln S_T$ (in limit)

- To proceed, we use $p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$ (i.e., risk-neutral probability). Here we use the Tylor series of e^x and also the fact $\Delta t \to 0$ as $n \to \infty$.[2]

$$p \approx \frac{1 + r\Delta t - (1 - \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2)}{(1 + \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2) - (1 - \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2)}$$
$$= \left(\frac{1}{2} + \frac{(r - \sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right)$$

- Plugging this p into $E(\ln S_T)$ in the previous page, the mean becomes

$$E(\ln S_T) = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T$$

where we use the fact $\Delta t = \frac{T}{n}$.

[2] $e^{r\Delta t} \approx 1 + r\Delta t + \frac{1}{2}r^2\Delta t^2$ (set Δt as x), $e^{-\sigma\sqrt{\Delta t}} \approx 1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t$ (set $\sqrt{\Delta t}$ as x).

Mean and Variance of $\ln S_T$ (in limit)

- The standard deviation of $\ln S_T$ is

$$\begin{aligned}\text{Std.Dev.}(\ln S_T) &= 2\sigma\sqrt{\Delta t} \times \sqrt{np(1-p)} \\ &= 2\sigma\sqrt{Tp(1-p)}.\end{aligned}$$

- Next, let's simplify the standard deviation. We find

$$\begin{aligned}p(1-p) &= \left(\frac{1}{2} + \frac{(r-\sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right)\left(\frac{1}{2} - \frac{(r-\sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right) \\ &= \frac{1}{4} - \frac{\left(r-\sigma^2/2\right)^2}{4\sigma^2}\Delta t \approx \frac{1}{4}\end{aligned}$$

- Thus, the standard deviation of $\ln S_T$ becomes

$$\text{Std.Dev.}(\ln S_T) = \sigma\sqrt{T}.$$

Log-Normal Property of Stock Prices

- Combining the mean and the standard deviation, we conclude

$$\ln S_T \sim \phi\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right)$$

  in the risk-neutral world.

Log-Normal Property of Stock Prices - Real probability

- Consider the real world where investors require the return α per annum on stock. Then, we can use the real probability p^* instead of p.

$$p^* = \frac{e^{\alpha \Delta t} - d}{u - d}$$

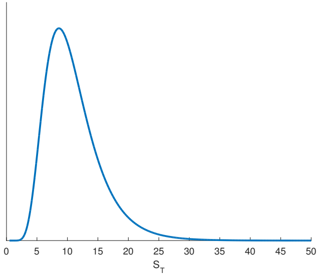
- Following the same logic as in the risk-neutral world, the real world distribution of stock price is

$$\ln S_T \sim \phi\left(\ln S_0 + \left(\alpha - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right)$$

Log-Normal Property of Stock Prices - Example

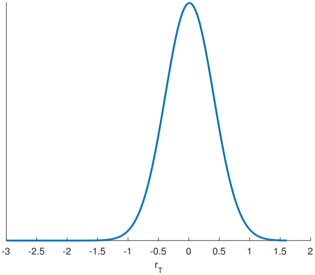
- Suppose that $S_0 = 10$, $r = 0.09$, $\sigma = 0.4$, and $T = 1$. Below are the probability density functions of S_T and r_T $(= \ln(S_T/S_0))$.

S_T



log-normal distribution

r_T



normal distribution

Probability of Option Exercise

- Using the distribution of future stock price under the risk-neutral measure, we can determine the probability of option exercise.

- Consider a European call with strike price K and expiration date T.

- What is the probability of option exercise,

$$\text{Prob}\left(S_T \geq K\right)$$

  when S_T is log-normally distributed?

Probability of Option Exercise

- The probability is ...

$$\begin{aligned}
&\text{Prob}\left(S_T \geq K\right) = \text{Prob}\left(\ln S_T \geq \ln K\right) \\
=&\text{Prob}\left(\frac{\ln S_T - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \geq \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&1 - \text{Prob}\left(\underbrace{\frac{\ln S_T - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}}_{\sim\phi(0,1)} < \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&1 - N\left(\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&N\left(\frac{-\ln K + \ln S_0 + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \equiv N(d_2)
\end{aligned}$$

where $\phi(0,1)$ is a standard normal random variable, and $N(x)$ is the cumulative distribution function of the standard normal.

Next

- Using the log-normal distribution of stock price, we can calculate the expected payoff of an option. This will lead us to the Black-Scholes-Merton formula.

- The exercise probability, $N(d_2)$, will be a part of the BSM result.

Math Review

Derivation of the BSM formula - Math Review

- In the derivation of the BSM formula, we need to compute the expected value of a function of random variable.
- This requires the understanding of a normal random variable and its probability density function.
- In addition, the calculation requires us to change variable in integration. This technique will be reviewed in the next slide.

Math Review - Normal Distribution

- Recall that to define $N(x)$, we consider a standard normal random variable Z.
- For a certain value x, $N(x)$ is the probability that Z is lower than or equal to x.

$$N(x) \equiv \text{Prob}(Z \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

- Graphically, $N(x)$ is the shadowed area in the below figure.

Figure 14.3 Shaded area represents $N(x)$.

- In Excel, we can use the function "norm.s.dist(x, TRUE)" to compute $N(x)$.

Math Review - Change of variable in integration

- Suppose that we integrate function $f(y)$ with respect to y:

$$\int f(y)dy.$$

- In addition, y is a function of another variable x, $y = g(x)$.

- Then, we can rewrite the above integration with respect to x

$$\int f(y)dy = \int f(g(x))g'(x)dx.$$

- Intuitively, we change dy to $g'(x)dx$ based on the derivative

$$\frac{dy}{dx} = g'(x)$$

Math Review - Change of variable in integration

e.g. Y is a normal random variable with mean m and the standard deviation w. $f(Y)$ is a function of the variable. Then, the expectation of $f(Y)$ is

$$E[f(Y)] = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(y-m)^2}{2w^2}} dy$$

- Consider a new variable $z = \frac{y-m}{w}$. Then, $y = m + wz$ and $(dy) = w(dz)$. We can rewrite the above integration in terms of z:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(y-m)^2}{2w^2}} dy &= \int_{-\infty}^{\infty} f(m+wz) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{z^2}{2}} w(dz) \\
&= \int_{-\infty}^{\infty} f(m+wz) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz
\end{aligned}$$

Derivation of Black-Scholes-Merton Model

Black-Scholes-Merton Model - Assumptions

- The derivation of BSM option price is based on following assumptions.
  - The stock price follows a log-normal distribution.
  - The risk-free rate, r, is constant and the same for all maturities.
  - There are no dividends during the life of the derivative.
  - There are no transaction costs or taxes.
  - There are no arbitrage opportunities.

Black-Scholes-Merton Model - Derivation

- Using the present-value approach, the call price is

$$c_0 = e^{-rT} E\left[\max(S_T - K, 0)\right].$$

  when we compute the expected payoff under the risk-neutral probability $\Rightarrow$ Risk-neutral valuation

- Utilizing the log-normal distribution of S_T, we can compute the expected option payoff. Then, by discounting as above, we obtain the option price.

Black-Scholes-Merton Model - Derivation

- First, let's calculate $E\left[\max(S_T - K, 0)\right]$

- Note that S_T is log-normally distributed in the risk-neutral world.

$$\ln S_T \sim \phi\left(\underbrace{\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T}_{\equiv m}, \underbrace{\sigma\sqrt{T}}_{\equiv w}\right)$$

- To simplify the notation, let V denote $\ln S_T$. So, $V \sim \phi(m, w)$. Then, the probability density function of V is

$$g(V) = \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(V-m)^2}{2w^2}}$$

- Let's use $g(V)$ to compute the expected payoff of the call.

Black-Scholes-Merton Model - Derivation

- The expected payoff is

$$
\begin{aligned}
E\left[\max(S_T - K, 0)\right] &= E\left[\max(e^V - K, 0)\right] \\
&= \int_{-\infty}^{\infty} \max(e^V - K, 0) g(V) dV \\
&= \int_{-\infty}^{\ln K} \underbrace{\max(e^V - K, 0)}_{=0} g(V) dV + \int_{\ln K}^{\infty} \underbrace{\max(e^V - K, 0)}_{=e^V - K} g(V) dV \\
&= \int_{\ln K}^{\infty} (e^V - K) g(V) dV \\
&= \underbrace{\int_{\ln K}^{\infty} e^V \cdot g(V) dV}_{\equiv \mathbb{A}} - \underbrace{\int_{\ln K}^{\infty} K \cdot g(V) dV}_{\equiv \mathbb{B}}
\end{aligned}
$$

- Let's calculate $\mathbb{A}$ and $\mathbb{B}$ separately and combine later.

Black-Scholes-Merton Model - Derivation

- Let's find $\mathbb{B}$ first.

$$\begin{aligned}
\mathbb{B} &= \int_{\ln K}^{\infty} K \cdot g(V)dV \\
&= K \int_{\ln K}^{\infty} g(V)dV \\
&= K \cdot \text{Prob}\,(V \geq \ln K) \\
&= K \cdot \text{Prob}\left(\underbrace{e^V}_{=S_T} \geq \underbrace{e^{\ln K}}_{=K}\right) \\
&= K \cdot N(d_2)
\end{aligned}$$

where $d_2 = \frac{\ln(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}}$.

Black-Scholes-Merton Model - Derivation

- Next, let's find $\mathbb{A}$.

$$\mathbb{A} = \int_{\ln K}^{\infty} e^V \cdot g(V) dV = \int_{\ln K}^{\infty} e^V \cdot \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(V-m)^2}{2w^2}} dV$$

- To simplify the calculation, define a new variable $Q = \frac{V-m}{w}$. Then, $V = m + wQ$, and $(dV) = w(dQ)$ in the change of variable in the integration.

$$\begin{aligned}
\mathbb{A} &= \int_{\frac{\ln K - m}{w}}^{\infty} e^{m+wQ} \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{Q^2}{2}} w \times dQ \\
&= \int_{\frac{\ln K - m}{w}}^{\infty} e^{m+wQ} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}} dQ \\
&= \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2} + wQ + m} dQ \\
&= \ldots
\end{aligned}$$

Black-Scholes-Merton Model - Derivation

$$\begin{aligned}
\mathbb{A} &= \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2} + wQ - \frac{w^2}{2} + \frac{w^2}{2} + m} dQ \\
&= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2} + wQ - \frac{w^2}{2}} dQ \\
&= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Q-w)^2}{2}} dQ
\end{aligned}$$

- To simplify, define a new variable $Y = Q - w$. Then, $Q = Y + w$ and $(dQ) = (dY)$ in the change of variable in the integration.

$$\begin{aligned}
\mathbb{A} &= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m - w^2}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Y^2}{2}} dY \\
&= e^{m + \frac{w^2}{2}} \times \text{Prob}\left(Y \geq \frac{\ln K - m - w^2}{w} \right) \\
&= \ldots
\end{aligned}$$

Black-Scholes-Merton Model - Derivation

$$
\begin{aligned}
\mathbb{A} &= e^{m+\frac{w^2}{2}} \times \left[1 - \text{Prob}\left(Y < \frac{\ln K - m - w^2}{w}\right)\right] \\
&= e^{m+\frac{w^2}{2}} \times \left[1 - N\left(\frac{\ln K - m - w^2}{w}\right)\right] \\
&= e^{m+\frac{w^2}{2}} \times N\left(\frac{-\ln K + m + w^2}{w}\right)
\end{aligned}
$$

Black-Scholes-Merton Model - Derivation

- In $\mathbb{A}$,

$$m + \frac{w^2}{2} = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T + \frac{\sigma^2 T}{2} = \ln S_0 + rT$$

$$\frac{-\ln K + m + w^2}{w} = \frac{\ln S_0 - \ln K + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

- Thus,

$$\begin{aligned}
\mathbb{A} &= e^{m + \frac{w^2}{2}} \times N\left(\frac{-\ln K + m + w^2}{w}\right) \\
&= S_0 e^{rT} \times N\left(\underbrace{\frac{\ln S_0 - \ln K + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}}_{\equiv d_1}\right) \\
&= S_0 e^{rT} \times N(d_1)
\end{aligned}$$

Black-Scholes-Merton Model - Derivation

- Now, let's combine $\mathbb{A}$ and $\mathbb{B}$.

$$\begin{aligned} E\left[\max(S_T - K, 0)\right] &= \mathbb{A} - \mathbb{B} \\ &= S_0 e^{rT} \times N(d_1) - K \times N(d_2) \end{aligned}$$

- The current price of the call is

$$\begin{aligned} c_0 &= e^{-rT} E\left[\max(S_T - K, 0)\right] \\ &= e^{-rT}\left[S_0 e^{rT} \times N(d_1) - K \times N(d_2)\right] \\ &= S_0 N(d_1) - K e^{-rT} N(d_2) \end{aligned}$$

Black-Scholes-Merton Model - Derivation

- Once the call option is obtained, we can easily drive the put price using the put-call parity.

$$
\begin{aligned}
p_0 &= c_0 + Ke^{-rT} - S_0 \\
&= S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT} - S_0 \\
&= -S_0 \left[1 - N(d_1)\right] + Ke^{-rT} \left[1 - N(d_2)\right] \\
&= -S_0 N(-d_1) + Ke^{-rT} N(-d_2)
\end{aligned}
$$

Black-Scholes-Merton Model

- The BSM model provides an analytic form that determines the option price as a function of the followings:
  - Current stock price S_0
  - Strike price K
  - Time to expiration T
  - Risk-free interest rate r
  - Volatility of underlying asset σ
- Through the BSM model, we can find the option price by simply inputting numbers into the option-pricing formula.

Black-Scholes-Merton Model - Result

- The prices of European call and put options on non-dividend-paying stock are

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and $N(x)$ is the cumulative probability distribution function for a standard normal random variable.

Black-Scholes-Merton Model - Example

Q. There is a 6-month European call option on a stock whose current price is \$42. The strike price is \$40, and the risk-free interest rate is 10% per annum. The stock volatility is 20% per annum. What is the price of the option?

Answer:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(42/40) + (0.1 + 0.2^2/2)(0.5)}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.6278$$

$$\begin{aligned} c &= S_0 N(d_1) - Ke^{-rT} N(d_2) \\ &= 42 \times N(0.7693) - 40e^{-0.1\times 0.5} \times N(0.6278) \\ &= 42 \times \text{norm.s.dist}(0.7693, \text{TRUE}) - 40e^{-0.1\times 0.5} \times \text{norm.s.dist}(0.6278, \text{TRUE}) \\ &= \$4.759. \end{aligned}$$

Black-Scholes-Merton Model - Example

- What if we use the binomial model for the previous question?
- Let's start with 10-step binomial model and increases the number of steps.

number of steps	option price
10	4.800
20	4.768
50	4.762
⋮	⋮
500	4.759
BSM price	4.759

- As the number of steps increases, the binomial price converges to the BSM price.

Black-Scholes-Merton Model – Example

Q. A European put option on a non-dividend-paying stock:

$$S_0 = \$60, \quad K = \$65, \quad T = 1 \text{ year}, \quad r = 5\% \text{ p.a.}, \ \sigma = 30\% \text{ p.a.}$$

What is the theoretical price of this put option under the BSM model?

Black-Scholes-Merton Model - Another Example

Q. Consider a derivative on a stock with the time to expiration T and the following payoff:

$$\begin{cases} 0 & \text{if } S_T < K_1 \\ K_1 & \text{if } K_1 \leq S_T < K_2 \\ 0 & \text{if } K_2 \leq S_T \end{cases}$$

where $K_2 > K_1$. What is the present value of the derivative? Provide an analytic expression of the price using $N(\cdot)$, the cumulative probability distribution function of a standard normal random variable.

Black-Scholes-Merton Model - Another Example

Answer: Let V denote $\ln S_T$. Then, V is normally distributed, i.e., $V \sim \phi(m, w)$. Let $g(V)$ denote the probability density function of V. To find the present value of the derivative, we first compute the expected option payoff:

$$
\begin{aligned}
E\left[\text{Payoff}\right] &= \int_{-\infty}^{\infty} \text{Payoff} \cdot g(V)dV \\
&= \int_{-\infty}^{\ln K_1} \text{Payoff} \cdot g(V)dV + \int_{\ln K_1}^{\ln K_2} \text{Payoff} \cdot g(V)dV \\
&\qquad + \int_{\ln K_2}^{\infty} \text{Payoff} \cdot g(V)dV \\
&= \int_{-\infty}^{\ln K_1} 0 \cdot g(V)dV + \int_{\ln K_1}^{\ln K_2} K_1 \cdot g(V)dV + \int_{\ln K_2}^{\infty} 0 \cdot g(V)dV \\
&= K_1 \int_{\ln K_1}^{\ln K_2} g(V)dV \\
&= K_1 \cdot \text{Prob}\left(\ln K_1 \leq V \leq \ln K_2\right) \\
&= K_1 \cdot \text{Prob}\left(K_1 \leq S_T \leq K_2\right) \\
&= K_1 \cdot \left[\text{Prob}\left(K_1 \leq S_T\right) - \text{Prob}\left(K_2 \leq S_T\right)\right]
\end{aligned}
$$

Black-Scholes-Merton Model - Another Example

Answer (cont'd):

$$= K_1 \cdot \left[N\left(\frac{\ln(S_0/K_1) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln(S_0/K_2) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) \right].$$

Next, multiplying by the discount factor, we obtain the present value as follows:

$$f_0 = e^{-rT} K_1 \cdot \left[N\left(\frac{\ln(S_0/K_1) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln(S_0/K_2) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \right) \right].$$

BSM Formula: Interpretation

- Under the Black–Scholes–Merton model, a call option can be viewed as being replicated by a portfolio of the underlying stock and a risk-free bond.

- In particular:

$$\Delta_c = \frac{\partial C}{\partial S} = N(d_1) > 0,$$

  meaning that $N(d_1)$ is the number of shares of stock held in the replicating portfolio for the call.

$$\Delta_p = \frac{\partial P}{\partial S} = -N(-d_1) < 0,$$

  meaning for a put the equivalent position is short stock.

- The term $K\,e^{-rT}\,N(d_2)$ represents the present-value of the amount borrowed (or short-bond position) in the replicating portfolio for a call.

- Hence the call price is simply the cost of the replicating portfolio at time 0:

$$c_0 \;=\; \Delta_c\,S_0 \;-\; B \;=\; S_0\,N(d_1) \;-\; K\,e^{-rT}\,N(d_2).$$

Extending the BSM model

Forward Formulation — One Formula for Everything

Rewrite the BSM call using the forward price $F_0 = S_0 e^{rT}$:

$$\boxed{c_0 = e^{-rT}\left[\, F_0\, N(d_1) - K\, N(d_2)\,\right]}$$

$$d_1 = \frac{\ln(F_0/K) + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

This same formula covers four cases. Just plug in the right F_0:

Underlying	Forward price F_0	Variant name
Non-dividend stock	$S_0\, e^{rT}$	Black-Scholes 1973
Continuous dividend	$S_0\, e^{(r-q)T}$	Merton 1973
Futures contract	F_0 (futures price)	Black '76
Currency	$S_0\, e^{(r-r_f)T}$	Garman-Kohlhagen 1983

Variant 1 — Dividend Stock (Merton)

- Suppose the underlying pays continuous dividend q .
  - Dividend should, for the purposes of option valuation, be defined as the reduction in the stock price.

- Replace the stock price S in the formula by Se^{-qT} [3]

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

, where $d_1 = \frac{\ln(S_0/K)+(r-q+\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. (called Merton model)

- Delta $= e^{-qT} N(d_1)$

- Put-Call parity: $p + S_0 e^{-qT} = c + Ke^{-rT}$

[3] For sketch of proof, go to the slide, "The BSM for dividend payout: Derivation".

Variant 2 — Options on Futures (Black '76)

- The underlying is a futures contract, so S in the equation is the futures price, call it F.
  - Remember $F_0 = S_0 e^{rT}$. As time passes, e^{rT} shrinks at the rate of r like dividend yield q. (assume Futures = Forward here).

- Replace the stock price S in the formula by the discounted value of the futures price F: Fe^{-rT}

$$c = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2) = e^{-rT}[FN(d_1) - KN(d_2)]$$

, where $d_1 = \frac{\ln(F/K)+(\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

- Delta $= e^{-rT}N(d_1)$

- Put-Call parity: $p + Fe^{-rT} = c + Ke^{-rT}$

Variant 3 — Currency Options (Garman-Kohlhagen)

- The price of the underlying is the exchange rate (in \$ per unit of FX). The underlying pays interest at the foreign riskless rate, so set $q = r_F$. The riskless rate r is the domestic rate (Garman-Kohlhagen Model).

- Replace the stock price S in the formula by $Se^{-r_F T}$

$$c = S_0 e^{-r_F T} N(d_1) - Ke^{-rT} N(d_2)$$

, where $d_1 = \frac{\ln(S_0/K)+(r-r_F+\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

- Delta $= e^{-r_F T} N(d_1)$

- Put-Call parity: $p + S_0 e^{-r_F T} = c + Ke^{-rT}$

Alternative Derivation I

Review

This derivation is also based on the Binomal Tree model in the risk-neutral world.

- The final stock price: $S_0\, u^j\, d^{\,n-j}$.
- The payoff from a European call option: $\max\bigl(S_0\, u^j\, d^{\,n-j} - K,\ 0\bigr)$.
- The probability of j upward and $n-j$ downward steps:

$$\frac{n!}{j!\,(n-j)!}\ p^j\,(1-p)^{n-j}$$

- The expected payoff:

$$\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\ p^j\,(1-p)^{n-j}\ \max\bigl(S_0\, u^j\, d^{\,n-j} - K,\ 0\bigr)$$

- The option value:

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\ p^j\,(1-p)^{n-j}\ \max\bigl(S_0\, u^j\, d^{\,n-j} - K,\ 0\bigr)$$

Review: Binomial Tree Derivation

We begin from the multi-step Binomial model in the risk-neutral world.

- Final stock price after n steps: $S_T(j) = S_0\, u^j\, d^{\,n-j}$.
- Payoff of a European call: $\max\bigl(S_0\, u^j\, d^{\,n-j} - K,\ 0\bigr)$.
- Probability of exactly j upward moves (and $n - j$ downward):

$$\Pr(j) = \frac{n!}{j!\,(n-j)!}\; p^j\,(1-p)^{\,n-j}$$

- Expected (risk-neutral) payoff:

$$\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\; p^j\,(1-p)^{\,n-j}\ \max\bigl(S_0\, u^j\, d^{\,n-j} - K,\ 0\bigr)$$

- Present value (call price):

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\; p^j\,(1-p)^{\,n-j}\ \max\bigl(S_0\, u^j\, d^{\,n-j} - K,\ 0\bigr)$$

Alternative Formulation of Call Price

Payoff positive if $S_0\, u^j\, d^{\,n-j} > K$, i.e.,

$$\ln\!\left(\tfrac{S_0}{K}\right) \;>\; -j\,\ln(u) \;-\; (n-j)\,\ln(d).$$

With $u = e^{\sigma\sqrt{T/n}}$ and $d = e^{-\sigma\sqrt{T/n}}$:

$$\ln\!\left(\tfrac{S_0}{K}\right) \;>\; n\,\sigma\sqrt{\tfrac{T}{n}} \;-\; 2\,j\,\sigma\sqrt{\tfrac{T}{n}} \quad\Longrightarrow\quad j \;>\; \frac{n}{2} \;-\; \frac{\ln(S_0/K)}{2\,\sigma\sqrt{T/n}}.$$

Thus

$$c = e^{-rT}\sum_{j>\alpha}\frac{n!}{j!\,(n-j)!}\;p^j\,(1-p)^{n-j}\;\max\!\big(S_0\,u^j\,d^{\,n-j}-K,\;0\big),\quad \alpha = \frac{n}{2}-\frac{\ln(S_0/K)}{2\,\sigma\sqrt{T/n}}.$$

Write $c = e^{-rT}\big(S_0\,U_1 - K\,U_2\big)$, with

$$U_1 = \sum_{j>\alpha}\frac{n!}{j!\,(n-j)!}\;p^j\,(1-p)^{n-j}\,u^j\,d^{\,n-j}, \qquad U_2 = \sum_{j>\alpha}\frac{n!}{j!\,(n-j)!}\;p^j\,(1-p)^{n-j}.$$

Increasing the Number of Steps: Convergence to BSM

As $n \to \infty$, $j \sim B(n,p) \longrightarrow \phi\big(np,\ \sqrt{np(1-p)}\big)$. Since $U_2 = \Pr(j > \alpha)$,

$$U_2 \ = \ \Pr\left(\frac{j - np}{\sqrt{np(1-p)}} \ > \ \frac{\alpha - np}{\sqrt{np(1-p)}} \right) \ = \ N\left(\frac{np - \alpha}{\sqrt{np(1-p)}} \right).$$

Substituting α:

$$U_2 \ = \ N\left(\frac{\ln(S_0/K)}{2\,\sigma\,\sqrt{T\,p(1-p)}} \ + \ \frac{\sqrt{n}\,(p - \frac{1}{2})}{\sqrt{p(1-p)}} \right).$$

Recall $p = \dfrac{e^{r\,T/n} - e^{-\sigma\sqrt{T/n}}}{e^{\sigma\sqrt{T/n}} - e^{-\sigma\sqrt{T/n}}}$, and by Taylor expansion $e^{rT/n} \approx 1 + r\,(T/n)$,
$e^{\pm\sigma\sqrt{T/n}} \approx 1 \pm \sigma\sqrt{T/n} + \frac{1}{2}\sigma^2(T/n)$.
Hence

$$p(1-p) \ \longrightarrow \ \tfrac{1}{4}, \qquad \sqrt{n}\,(p - \tfrac{1}{2}) \ \longrightarrow \ \frac{(r - \frac{1}{2}\sigma^2)\sqrt{T}}{2\,\sigma}.$$

Therefore

$$U_2 \ = \ N\left(\frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)\,T}{\sigma\,\sqrt{T}} \right).$$

Final Step: From Binomial to Black–Scholes

Starting from

$$U_1 \;=\; \sum_{j>\alpha} \frac{n!}{j!\,(n-j)!}\;(u\,p)^j\,\big(d\,(1-p)\big)^{\,n-j},$$

let

$$p^* \;=\; \frac{p\,u}{p\,u+(1-p)\,d}, \qquad 1-p^* \;=\; \frac{(1-p)\,d}{p\,u+(1-p)\,d}.$$

Then

$$U_1 \;=\; \big(p\,u+(1-p)\,d\big)^{\,n} \sum_{j>\alpha} \frac{n!}{j!\,(n-j)!}\;(p^*)^j\,(1-p^*)^{\,n-j}.$$

Because $p\,u+(1-p)\,d = e^{rT}$,

$$U_1 \;=\; e^{rT} \sum_{j>\alpha} \frac{n!}{j!\,(n-j)!}\;(p^*)^j\,(1-p^*)^{\,n-j}.$$

So in the limit as $n \to \infty$,

$$U_1 \;=\; e^{rT}\,N\!\left(\frac{\ln(S_0/K)+(r+\frac{1}{2}\sigma^2)\,T}{\sigma\sqrt{T}}\right).$$

$$\boxed{c \;=\; S_0\,N(d_1) \;-\; K\,e^{-rT}\,N(d_2).}$$

Alternative Derivation II

Why a second derivation?

- **Binomial route (Derivation I):** build a replicating portfolio at each node, take the limit $n \to \infty$, recover BSM. *Discrete and constructive.*
- **Continuous route (Derivation II):** model the stock as a continuous random process from the start, hedge it dynamically, derive a PDE. *Continuous and analytical.*
- Both rest on the same idea — *a riskless arbitrage-free portfolio must earn r* — but the continuous route requires new mathematics: **Itô (stochastic) calculus**.

Goal of this section: build the minimum stochastic calculus needed to derive the BSM PDE — with intuition first, formalism second.

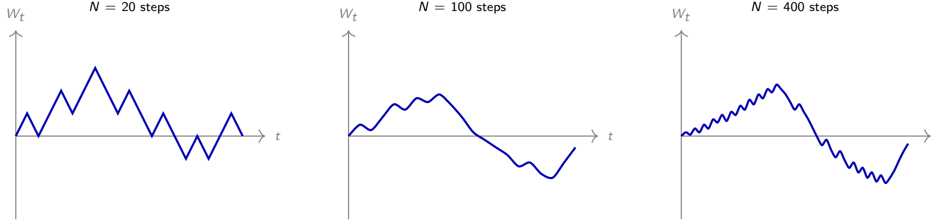
Underlying assumptions of the BSM model

- Options are European.
- "Perfect" markets — no transaction costs, no taxes, no short-sale constraints, no indivisibilities.
- No limits on borrowing or lending at a known risk-free rate r.
- The price of the underlying follows a **lognormal diffusion** process.
- The return volatility σ of the underlying is known and constant.
- No dividends or cash payouts prior to maturity.

Every assumption is wrong in some way.

From random walks to Brownian motion

A simple random walk: at each step, move $\pm\Delta x$ with probability $1/2$. Now *shrink the time step* $\Delta t \to 0$ *while scaling* $\Delta x = \sqrt{\Delta t}$.



- As $N \to \infty$ with $\Delta x = \sqrt{\Delta t}$, the path becomes continuous but *nowhere differentiable*.
- This limit is a **standard Brownian motion** W_t. Variance scales linearly: $\mathrm{Var}(W_t) = t$.

Brownian motion — the formal definition

A process $\{W_t : t \geq 0\}$ is a **Brownian motion** (Wiener process) if:

1. $W_0 = 0$.
2. It has continuous paths.
3. For $0 \leq s < t$, the increment $W_t - W_s$ is independent of the past and distributed $\mathcal{N}(0,\ t-s)$.

Informal shorthand (useful for calculations):

$$dW_t \;\approx\; \epsilon\sqrt{dt}, \qquad \epsilon \sim \mathcal{N}(0,1).$$

Why this matters for us: every shock in our model will be built from dW_t. The crucial property — and the one that breaks ordinary calculus — is that $(dW_t)^2 = dt$ on average, not zero.

Numerical check: if $\Delta t = 0.01$ and $\epsilon = 1.5$, then the shock is $1.5 \times \sqrt{0.01} = 0.15$.

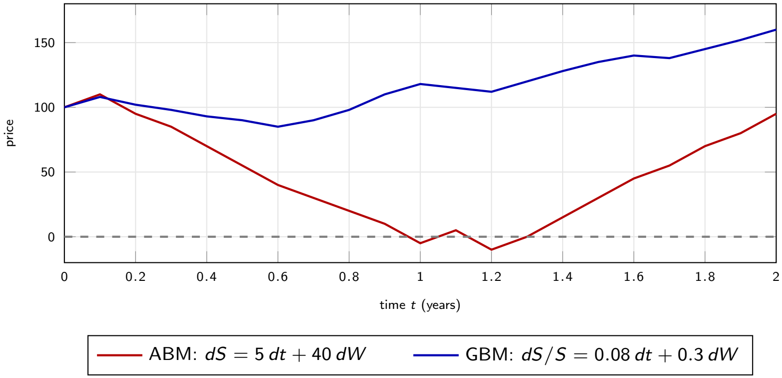
Building dynamics for the stock price

Three candidate models. Which one captures real stock behavior?

- **Candidate 1.** $dS_t = \mu\, dt$ (pure drift, no noise)
  $\Rightarrow S_t = S_0 + \mu t$. Deterministic. Unrealistic.
- **Candidate 2.** $dS_t = \mu\, dt + \sigma\, dW_t$ (arithmetic Brownian motion)
  $\Rightarrow$ Stock can become *negative*. Violates limited liability.
- **Candidate 3.** $\dfrac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$ (geometric Brownian motion)
  Drift and noise are *proportional to price*. S_t stays positive. **This is the BSM model.**

For a risk-free asset ($\sigma = 0$, $\mu = r$): $S_t = S_0 e^{rt}$. The model nests the deterministic bond as a special case.

Why GBM?



- **ABM (red)** can and does cross zero — impossible for a stock with limited liability.
- **GBM (blue)** stays strictly positive; volatility scales with price; $\ln S_t$ is normal.

Generalizations: Wiener and Itô processes

- **Generalized Wiener process:** constant drift, constant volatility.

$$dX_t = \mu\, dt + \sigma\, dW_t \qquad (\mu, \sigma \text{ constant})$$

- **Itô process:** drift and volatility may depend on state and time.

$$dX_t = a(X_t, t)\, dt + b(X_t, t)\, dW_t$$

- **Geometric Brownian motion** is an Itô process with $a(S, t) = \mu S$, $b(S, t) = \sigma S$:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t.$$

Every model we use — BSM, Heston, CIR, SABR — is an Itô process. The framework is general; what changes are the choices of $a(\cdot)$ and $b(\cdot)$.

Other stochastic processes you'll meet

Process	Equation	Used for
Arithmetic BM	$dX_t = \mu\, dt + \sigma\, dW_t$	idealized prototype
Geometric BM	$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$	stock prices (BSM)
Ornstein–Uhlenbeck	$dY_t = \kappa(\theta - Y_t)\, dt + \sigma\, dW_t$	interest rates, spreads (mean reversion)
Cox–Ingersoll–Ross	$dr_t = \kappa(\theta - r_t)\, dt + \sigma\sqrt{r_t}\, dW_t$	short rates (stays ≥ 0)
Heston (vol)	$dv_t = \kappa(\theta - v_t)\, dt + \xi\sqrt{v_t}\, dW_t$	stochastic volatility models

- **Common structure:** drift term + noise term scaled by some function of state.
- **Mean reversion** appears in rates and volatility (which empirically don't trend forever). **Square-root diffusion** keeps the process non-negative.
- For BSM today we use *only* GBM. The others are for later courses (fixed-income, exotic options).

Itô's lemma — the chain rule, plus a correction

If X_t is an Itô process $dX_t = a\,dt + b\,dW_t$ and $G = G(X_t, t)$ is twice-differentiable, then

$$dG \;=\; \underbrace{\left(\frac{\partial G}{\partial X}a + \frac{\partial G}{\partial t} + \tfrac{1}{2}\frac{\partial^2 G}{\partial X^2}b^2\right)dt}_{\text{drift, including the } b^2 \text{ correction}} \;+\; \underbrace{\frac{\partial G}{\partial X}b\,dW_t}_{\text{noise}}\,.$$

Where does this come from? A Taylor expansion of G:

$$dG \approx \frac{\partial G}{\partial X}dX + \frac{\partial G}{\partial t}dt + \tfrac{1}{2}\frac{\partial^2 G}{\partial X^2}(dX)^2 + \cdots$$

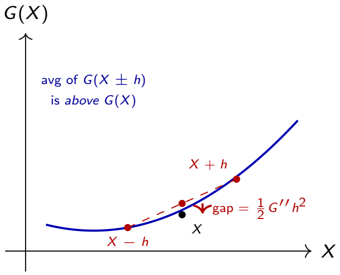
In ordinary calculus, $(dX)^2$ is negligible. In Itô calculus it is *not*:

$$(dX)^2 = (a\,dt + b\,dW_t)^2 = b^2\,(dW_t)^2 + \text{higher order} = b^2\,dt.$$

This single rule — $(dW_t)^2 = dt$ — is what makes stochastic calculus different from ordinary calculus. It is responsible for every "surprise" term that follows.

Why the convexity term appears — intuition

- Take a **convex** function G (e.g., $G(x) = x^2$).
- Suppose X jitters symmetrically: equally likely to move up by h or down by h.
- For a *linear* function, the up- and down-moves cancel in expectation.
- For a *convex* function, the up-move gains more than the down-move loses.
- So even though $\mathbb{E}[dX] = 0$, we have $\mathbb{E}[dG] > 0$. Curvature *creates* drift.



The Itô correction $\frac{1}{2} G'' b^2\, dt$ is exactly this gap, in instantaneous form. Jensen's inequality, made dynamic.

Applying Itô's lemma to $\ln S_t$

Stock: $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$. Let $G(S_t) = \ln S_t$. Then

$$\frac{\partial G}{\partial S} = \frac{1}{S_t}, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S_t^2}, \quad \frac{\partial G}{\partial t} = 0.$$

Plug into Itô's lemma:

$$\begin{aligned} d(\ln S_t) &= \left[\tfrac{1}{S_t} \cdot \mu S_t + 0 + \tfrac{1}{2} \cdot \left(-\tfrac{1}{S_t^2}\right) \cdot \sigma^2 S_t^2\right] dt + \tfrac{1}{S_t} \cdot \sigma S_t \, dW_t \\ &= \left(\mu - \tfrac{\sigma^2}{2}\right) dt + \sigma \, dW_t. \end{aligned}$$

The drift of $\ln S_t$ is $\mu - \frac{\sigma^2}{2}$, not μ. The missing $-\frac{\sigma^2}{2}$ is the convexity correction from the previous slide — the price you pay (in log-units) for volatility.

Consequence: S_T is lognormal

Integrating the previous equation from 0 to T:

$$\ln S_T - \ln S_0 \;=\; \left(\mu - \tfrac{\sigma^2}{2}\right) T + \sigma\, W_T \;\sim\; \mathcal{N}\!\left(\left(\mu - \tfrac{\sigma^2}{2}\right) T,\; \sigma^2\, T\right).$$

Equivalently,

$$\boxed{S_T = S_0 \exp\!\left[\left(\mu - \tfrac{\sigma^2}{2}\right) T + \sigma\, W_T\right]}$$

- $\ln S_T$ is normal $\Rightarrow$ S_T is **lognormal** $\Rightarrow$ $S_T > 0$ always.
- Mean of S_T: $\mathbb{E}[S_T] = S_0 e^{\mu T}$. The $-\frac{\sigma^2}{2}$ cancels in expectation (lognormal moment formula).
- Median is below mean — lognormal distributions are right-skewed.

This is the distribution under which we will integrate the call payoff. The integral closes in closed form $\Rightarrow$ the BSM formula.

The hedging trick — killing the randomness

Let $V = V(S_t, t)$ be the value of any derivative on S_t. Itô's lemma gives:

$$dV = \left(\frac{\partial V}{\partial S} \mu S_t + \frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S_t^2 \right) dt + \frac{\partial V}{\partial S} \sigma S_t \, dW_t.$$

Construct a portfolio: long 1 derivative, short $\Delta = \partial V / \partial S$ shares.

$$\Pi \; = \; V - \frac{\partial V}{\partial S} \, S_t.$$

Why this choice of Δ? Compute $d\Pi$:

$$d\Pi = dV - \frac{\partial V}{\partial S} \, dS_t = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sigma^2 S_t^2 \right) dt.$$

The dW_t term *cancels exactly*. The hedged portfolio is locally riskless — by construction.

From riskless return to the BSM PDE

Since Π is riskless over dt, no-arbitrage forces $d\Pi = r\Pi\, dt$:

$$\left(\tfrac{\partial V}{\partial t} + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2\right) dt = r\big(V - \tfrac{\partial V}{\partial S} S_t\big)\, dt.$$

Rearrange:

$$\boxed{\ \frac{\partial V}{\partial t} + r\, S\, \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} \ =\ r\, V.\ }$$

- This is the **Black–Scholes–Merton PDE**. Every European derivative on S_t (call, put, digital, forward) satisfies it.
- The contract type appears *only* in the terminal condition (e.g., $V(S,T) = \max(S - K, 0)$ for a call).
- For American puts on non-dividend stocks, no closed-form solution exists; the PDE is solved numerically.

The deep magic: where did μ go?

Look at the PDE again:

$$\frac{\partial V}{\partial t} + r\,S\,\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} = r\,V.$$

The stock's expected return μ does not appear. Only r, σ, and S do.

- Started with $dS_t = \mu S_t dt + \sigma S_t dW_t$. The hedging step canceled the dW_t term, which dragged μ out with it.
- **Why this is profound:** two investors can disagree about μ (the bull thinks 15%, the bear thinks 2%) and still agree on the option price.
- All that matters is **how much the stock jitters** (σ), not which direction it drifts.
- Equivalent to saying: price options *as if* the stock drifted at r in a fictitious "risk-neutral" world. This is the foundation of all modern derivative pricing.

Verifying the PDE on three simple contracts

$$\text{PDE:}\ \ \partial_t V + rS\,\partial_S V + \tfrac{1}{2}\sigma^2 S^2 \partial_{SS} V = rV.$$

Payoff at T	$V(S,t)$	$\partial_t V$	$\partial_S V$	$\partial_{SS} V$	Plug into PDE
Stock S_T	S_t	0	1	0	$rS = r\,V$ ✓
Constant K	$Ke^{-r(T-t)}$	rV	0	0	$rV = rV$ ✓
Forward $S_T - K$	$S_t - Ke^{-r(T-t)}$	$-rKe^{-r(T-t)}$	1	0	$-rKe^{-r(T-t)} + rS = rV$ ✓
Call (BSM)	$S_t N(d_1) - Ke^{-r(T-t)} N(d_2)$	(lengthy)	$N(d_1)$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	holds

- Every European derivative is a different boundary condition on the same PDE.
- The call formula's PDE check is mechanical but tedious; the key identity is $S_0 N'(d_1) = Ke^{-rT} N'(d_2)$.

The Black–Scholes PDE – Verification of Special Cases

$$\frac{\partial V}{\partial t} + r\,S\,\frac{\partial V}{\partial S} + \tfrac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} = r\,V$$

- If $V(S,T) = S_T$, i.e., the underlying stock itself $\rightarrow V(S,t) = S_t$.
  - Then $\frac{\partial V}{\partial t} = 0$, $\frac{\partial V}{\partial S} = 1$, $\frac{\partial^2 V}{\partial S^2} = 0$.
  - Substituting: $0 + r\,S \cdot 1 + \frac{1}{2}\sigma^2 S^2 \cdot 0 = r\,S = r\,V$.
  - So the PDE holds.

- If $V(S,T) = K$, a constant payoff $\rightarrow V(S,t) = K\,e^{-r\,(T-t)}$.
  - Then $\frac{\partial V}{\partial S} = 0$, $\frac{\partial^2 V}{\partial S^2} = 0$, and $\frac{\partial V}{\partial t} = r\,K\,e^{-r\,(T-t)} = r\,V$.
  - Substituting: $r\,V + r\,S \cdot 0 + 0 = r\,V$.
  - The PDE is satisfied.

- If $V(S,T) = S_T - K$ (a forward payoff) $\rightarrow V(S,t) = S_t - K\,e^{-r\,(T-t)}$.
  - Then $\frac{\partial V}{\partial S} = 1$, $\frac{\partial^2 V}{\partial S^2} = 0$, $\frac{\partial V}{\partial t} = -r\,K\,e^{-r\,(T-t)}$.
  - Left side: $-r\,K\,e^{-r\,(T-t)} + r\,S \cdot 1 + 0 = r\,V$.
  - Again the PDE holds.

Verification that the Call Price Satisfies the PDE

- It also holds for a European option on a non-dividend-paying stock. It's more complicated to verify, though.
- The PDE is extremely general. What changes between contracts (stock, bond, forward, option) is the *terminal condition* (and any boundary conditions). Once you know the terminal condition, you pick the corresponding solution that satisfies the PDE. Refer to standard derivations.

Where this leaves us

- **Two derivations, one answer.** The binomial route (Derivation I) and the Itô route (Derivation II) deliver the same BSM call formula. They are two windows onto the same no-arbitrage principle.
- **Stochastic calculus in one sentence:** ordinary Taylor expansion plus the rule $(dW_t)^2 = dt$.
- **Itô's lemma in one sentence:** the chain rule, with a convexity correction proportional to the second derivative.
- **The BSM PDE in one sentence:** dynamically hedge the noise away, then no-arbitrage pins down the drift.
- **Punchline:** μ disappears. Option prices depend only on r, σ, and the contract's payoff structure — not on the trader's view of where the stock is going.

Summary — BSM in One Page

1. **Risk-neutral pricing:** $V_0 = e^{-rT} E^{\mathbb{Q}}[\text{payoff}]$. BSM is the GBM special case.
2. **Log-normal property:** $\ln S_T \sim \phi(\ln S_0 + (r - \frac{\sigma^2}{2})T, \sigma\sqrt{T})$. The $\sigma^2/2$ is Jensen's inequality.
3. **Call price:** $c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$, put by parity.
4. **Interpretation:** $N(d_2) = \mathbb{Q}(\text{ITM})$; $N(d_1) = \Delta$, the share count in the replicating portfolio.
5. μ **cancels.** Hedging removes preferences; everyone agrees on the price given inputs.
6. **One formula, many variants.** $c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$ covers stock, dividend, futures, FX.
7. **PDE form.** $V_t + rS\, V_S + \frac{1}{2}\sigma^2 S^2\, V_{SS} = rV$, with contract-specific terminal condition.
8. σ **is the only input you don't observe.** Lec 12 turns this around: imply σ from quotes.

Appendix 1: The BSM for dividend payout

BSM with Continuous Dividend Yield: Derivation (1)

- Suppose the stock pays a continuous dividend yield q. Then, during dt, the stockholder receives a dividend

$$dD = q\,S\,\frac{\partial V}{\partial S}\,dt.$$

- The change in the value of the hedged portfolio is the sum of the change in portfolio value and the dividend income:

$$dW_t = d\Pi + dD.$$

- Using Itô's Lemma and the hedge ratio $\frac{\partial V}{\partial S}$, we have:

$$dW_t = \left(-\frac{\partial V}{\partial t} - \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + qS\frac{\partial V}{\partial S}\right)dt.$$

- Since the portfolio is instantaneously riskless, it must earn the risk-free rate r:

$$dW_t = r\,\Pi\,dt = r\left(-V + S\frac{\partial V}{\partial S}\right)dt.$$

BSM with Continuous Dividend Yield: Derivation (2)

- Equating the two expressions for dW_t and rearranging gives:

$$\frac{\partial V}{\partial t} + (r - q)S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.$$

- This is the **Black–Scholes–Merton PDE with dividends.** The dividend yield q reduces the drift of the stock under the risk-neutral measure.

- The corresponding risk-neutral stock price process is:

$$dS = (r - q)S\,dt + \sigma S\,dz.$$

- For a European call, solving the PDE gives the **Black–Scholes formula with dividends:**

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2),$$

  where

$$d_{1,2} = \frac{\ln(S_0/K) + (r - q \pm \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Appendix 2: From BSM PDE to BSM equation

Step 1: The Black–Scholes PDE

- We start with the partial differential equation (PDE) for the option value $V(S, t)$:
$$\frac{\partial V}{\partial t} + r\,S\,\frac{\partial V}{\partial S} + \tfrac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} \;=\; r\,V.$$
- Here:
    - S = underlying stock price at time t.
    - r = risk-free interest rate (continuous).
    - σ = volatility of the stock's returns.
    - The terminal (boundary) condition is:
    $$V(S, T) = \max(S - K,\, 0),$$
    for a European call option with strike K and maturity T.
- This PDE comes from hedging + Itô's Lemma + no-arbitrage.

Step 2: Change of Variables

- Solving the PDE directly is hard, so we perform a change of variables to simplify it. Typical transformations include:
  - $\tau = T - t$ (time to maturity).
  - $x = \ln(S/K)$ (log-stock variable).
  - Introduce a new function $u(x, \tau) = e^{r\tau} V(S, t)$ so that the discount-term rV disappears.
- Under these changes, the PDE is transformed into a "heat equation" form (a simpler diffusion PDE), for which standard solutions are known.
- This step is therefore a mathematical trick to make the PDE solvable with known methods.

Step 3: Solve the Transformed PDE

- Once in the "heat-equation" form, one applies known solution methods (e.g., separation of variables, Green's functions) to find $u(x, \tau)$.
- Then we revert the change of variables:

$$V(S, t) = e^{-r(T-t)}\, u\big(\ln(S/K),\ T - t\big).$$

- The result is an expression involving the normal cumulative distribution function $N(\cdot)$.
- In returning to the original variables, we obtain the closed-form formula for a European call option:

$$C = S\, N(d_1) \ -\ K\, e^{-r(T-t)}\, N(d_2),$$

with

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\,\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}.$$

Step 4: Interpretation & Key Insights

- Notice that the expected stock return μ does not appear in the final formula — only the risk-free rate r and volatility σ.
- Why? Because of risk-neutral valuation: in a hedged portfolio the expected return of the underlying becomes irrelevant.
- The formula therefore = discounted expected payoff under the "risk-neutral measure".