BUSS386 Problem Set 9 — Solutions

Binomial Trees

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Problem 1 — One-step binomial replication

$u = 60/50 = 1.20$, $d = 45/50 = 0.90$, $T = 0.5$, $r = 4\%$.

(a) $p = (e^{0.04 \cdot 0.5} - 0.90)/(1.20 - 0.90) = (1.0202 - 0.90)/0.30 = \boxed{0.4040}$.

(b) Call payoffs: $f_u = \max(60 - 50, 0) = 10$, $f_d = 0$.
$\Delta = (10 - 0)/(60 - 45) = 10/15 = \boxed{0.6667}$.
$B = e^{-0.02}(1.20 \cdot 0 - 0.90 \cdot 10)/0.30 = 0.9802 \cdot (-30) = -29.41$ (borrow).
$f_0 = 0.6667 \cdot 50 - 29.41 = 33.33 - 29.41 = \boxed{\$3.93}$.

(c) RN formula: $f_0 = e^{-0.02}[0.4040 \cdot 10 + 0.5960 \cdot 0] = 0.9802 \cdot 4.04 = \$3.96$ (matches within rounding).

(d) Put: $f_u = 0$, $f_d = \max(50-45, 0) = 5$. $p_0 = e^{-0.02}[0.4040 \cdot 0 + 0.5960 \cdot 5] = 0.9802 \cdot 2.98 = \boxed{\$2.92}$.
Parity check: $c_0 + Ke^{-rT} = 3.93 + 50 \cdot 0.9802 = 53.04$; $\quad p_0 + S_0 = 2.92 + 50 = 52.92$. Matches within rounding. ✓

Problem 2 — Two-step European put on KOSPI 200

$S_0 = 360$, $u = 1.05$, $d = 0.95$, $\Delta t = 0.25$, $r = 3\%$, $K = 360$.

(a) $p = (e^{0.0075} - 0.95)/(1.05 - 0.95) = (1.00753 - 0.95)/0.10 = \boxed{0.5753}$.

(b) Terminal index levels and put payoffs:

Node	S_T	$f_T = \max(360 - S_T, 0)$
uu	$360 \cdot 1.1025 = 396.90$	0
ud	$360 \cdot 0.9975 = 359.10$	0.90
dd	$360 \cdot 0.9025 = 324.90$	35.10

(c) $p^2 = 0.3310$, $2p(1-p) = 0.4886$, $(1-p)^2 = 0.1804$.
$E^{\mathbb{Q}}[\text{payoff}] = 0.3310 \cdot 0 + 0.4886 \cdot 0.90 + 0.1804 \cdot 35.10 = 0.440 + 6.332 = 6.772$.
$f_0 = e^{-0.015} \cdot 6.772 = 0.9851 \cdot 6.772 = \boxed{6.67}$ index pts.

(d) Per contract $= 6.67 \cdot$ ₩250,000 $=$ $\boxed{\text{₩}1{,}667{,}500}$.

Problem 3 — Three-step European call

$S_0 = K = 30$, $u = 1.1$, $d = 0.9$, $r = 5\%$, $N = 3$.

(a) $p = (e^{0.05} - 0.9)/0.2 = \boxed{0.7564}$.
Terminal payoffs:

j ups	$S_T = 30 \cdot 1.1^j \cdot 0.9^{3-j}$	$f_j = \max(S_T - 30, 0)$	$\binom{3}{j} p^j (1-p)^{3-j}$
0	21.87	0	0.0145
1	26.73	0	0.1349
2	32.67	2.67	0.4185
3	39.93	9.93	0.4321

(b) $E^{\mathbb{Q}}[\text{payoff}] = 0.4185 \cdot 2.67 + 0.4321 \cdot 9.93 = 1.117 + 4.291 = 5.408$.
$f_0 = e^{-0.15} \cdot 5.408 = 0.8607 \cdot 5.408 = \boxed{\$4.65}$.

(c) Backward induction confirms (step 2: $f_{uu} = 9.93$, $f_{ud} = 2.67$, $f_{dd} = 0$ at j=3,2,1 respectively, but applied to step-2 nodes):

- Step 2 nodes ($S = 36.30, 29.70, 24.30$): backward step gives 7.99, 1.92, 0.
- Step 1 nodes ($S = 33.0, 27.0$): 6.06, 1.40.
- Root: $e^{-0.05}[0.7564 \cdot 6.06 + 0.2436 \cdot 1.40] = 0.9512 \cdot (4.583 + 0.341) = 0.9512 \cdot 4.924 = \boxed{\$4.68}$ (closed-form and induction agree within rounding).

Problem 4 — American put

(a) Terminal put payoffs: $f_0 = \max(30 - 21.87, 0) = 8.13$, $f_1 = 3.27$, $f_2 = 0$, $f_3 = 0$.

(b) Backward induction with early-exercise check:

- **Step 2.** $S = 36.30$: hold $= 0$, intrinsic $= 0 \Rightarrow 0$. $S = 29.70$: hold $= e^{-0.05}[0.7564 \cdot 0 + 0.2436 \cdot 3.27] = 0.758$, intrinsic $= 0.30 \Rightarrow$ hold, 0.758. $S = 24.30$: hold $= e^{-0.05}[0.7564 \cdot 3.27 + 0.2436 \cdot 8.13] = 0.9512 \cdot 4.453 = 4.236$, intrinsic $= 5.70 \Rightarrow \boxed{\text{exercise}}$, 5.70.
- **Step 1.** $S = 33.0$: hold $= e^{-0.05}[0.7564 \cdot 0 + 0.2436 \cdot 0.758] = 0.176$, intrinsic $= 0 \Rightarrow$ hold, 0.176. $S = 27.0$: hold $= e^{-0.05}[0.7564 \cdot 0.758 + 0.2436 \cdot 5.70] = 0.9512 \cdot 1.962 = 1.866$, intrinsic $= 3.00 \Rightarrow \boxed{\text{exercise}}$, 3.00.
- **Root.** hold $= e^{-0.05}[0.7564 \cdot 0.176 + 0.2436 \cdot 3.00] = 0.9512 \cdot (0.133 + 0.731) = 0.822$, intrinsic $= 0 \Rightarrow$ hold.

(c) $f_0^{\text{Am}} = \boxed{\$0.82}$.

(d) European put: same backward induction without exercise check. Down-down step-2 value becomes 4.236 (not 5.70). Step 1 down $= e^{-0.05}[0.7564 \cdot 0.758 + 0.2436 \cdot 4.236] = 0.9512 \cdot (0.573 + 1.032) = 1.527$. Root $= e^{-0.05}[0.7564 \cdot 0.176 + 0.2436 \cdot 1.527] = 0.9512 \cdot (0.133 + 0.372) = \boxed{\$0.48}$.
Early-exercise premium $= 0.82 - 0.48 = \boxed{\$0.34}$ (a 71% uplift over the European).

Problem 5 — American call with dividend

(a) Step-1 cum-div: $S_u^{\text{cum}} = 120$, $S_d^{\text{cum}} = 85$. Ex-div: $S_u^{\text{ex}} = 117$, $S_d^{\text{ex}} = 82$.
Terminal: $S_{uu} = 117 \cdot 1.20 = 140.4$, $S_{ud} = 117 \cdot 0.85 = 99.45 = 82 \cdot 1.20 + \epsilon$, $S_{dd} = 82 \cdot 0.85 = 69.70$.

(b) $p = (e^{0.02} - 0.85)/0.35 = (1.0202 - 0.85)/0.35 = \boxed{0.4863}$.
$f_{uu} = 40.4$, $f_{ud} = \max(99.45 - 100, 0) = 0$, $f_{dd} = 0$.

(c) Up node, $S = 120$ cum-div:

- Exercise now: $120 - 100 = 20$, captures dividend.
- Hold (after div drop to 117): $e^{-0.02}[0.4863 \cdot 40.4 + 0.5137 \cdot 0] = 0.9802 \cdot 19.65 = 19.26$.

$\Rightarrow$ $\boxed{\text{Exercise early}}$, $f_u = 20$.
Down node, $S = 85$ cum-div: exercise $= -15 < 0$; hold $= 0$. $\Rightarrow f_d = 0$.

(d) Root: hold $= e^{-0.02}[0.4863 \cdot 20 + 0] = 0.9802 \cdot 9.726 = 9.535$. Intrinsic $= 0$. $\Rightarrow f_0 = \boxed{\$9.54}$.

(e) The dividend transfers value out of the stock into the dividend stream, which an unexercised call does not receive. When the dividend exceeds the interest forgone on K plus the put-side time value, exercising captures it.

Problem 6 — Calibration

$\sigma = 0.25$, $\Delta t = 0.25$, $r = 0.04$, $q = 0.01$, $S_0 = 100$, $K = 100$, $N = 4$.

(a) $u = e^{0.25 \cdot 0.5} = e^{0.125} = \boxed{1.1331}$.
$d = 1/u = \boxed{0.8825}$.
$p = (e^{(0.04-0.01)\cdot 0.25} - 0.8825)/(1.1331 - 0.8825) = (e^{0.0075} - 0.8825)/0.2506 = (1.00753 - 0.8825)/0.2506 = \boxed{0.4990}$.

(b) Terminal prices ($S_T = 100 \cdot u^j \cdot d^{4-j}$):

j	S_T	$f_j = \max(S_T - 100, 0)$
0	60.65	0
1	77.88	0
2	100.00	0
3	128.41	28.41
4	164.87	64.87

(c) $\binom{4}{3} p^3 (1-p)^1 = 4 \cdot 0.1243 \cdot 0.5010 = 0.2491$.
$\binom{4}{4} p^4 = 0.0620$.
$E^{\mathbb{Q}}[\text{payoff}] = 0.2491 \cdot 28.41 + 0.0620 \cdot 64.87 = 7.076 + 4.022 = 11.10$.
$f_0 = e^{-0.04 \cdot 1} \cdot 11.10 = 0.9608 \cdot 11.10 = \boxed{\$10.66}$.

Problem 7 — Merton model

$V_0 = ₩\,500\text{B}$, $\sigma_V = 0.20$, $L = ₩\,400\text{B}$, $T = 2$, $r = 0.03$, $u = e^{0.20} = 1.2214$, $d = e^{-0.20} = 0.8187$.

(a) Terminal:

Node	V_T (₩B)	$E_T = \max(V_T - 400, 0)$
uu	$500 \cdot e^{0.40} = 745.91$	345.91
ud	$500 \cdot e^{0} = 500.00$	100.00
dd	$500 \cdot e^{-0.40} = 335.16$	0

(b) $p = (e^{0.03} - 0.8187)/(1.2214 - 0.8187) = (1.03045 - 0.8187)/0.4027 = \boxed{0.5258}$.
$p^2 = 0.2765$, $2p(1-p) = 0.4990$, $(1-p)^2 = 0.2245$.
$E_0 = e^{-0.06}[0.2765 \cdot 345.91 + 0.4990 \cdot 100 + 0.2245 \cdot 0]$
$= 0.9418 \cdot (95.64 + 49.90) = 0.9418 \cdot 145.54 = \boxed{₩\,137.06\text{B}}$.

(c) $D_0 = V_0 - E_0 = 500 - 137.06 = \boxed{₩\,362.94\text{B}}$.
Implied yield y: $400 \cdot e^{-2y} = 362.94 \Rightarrow y = -\frac{1}{2}\ln(362.94/400) = -\frac{1}{2}\ln(0.9074) = \boxed{4.86\%}$.

(d) Credit spread $= 4.86\% - 3.00\% = \boxed{186 \text{ bp}}$.

(e) Korean application: pricing chaebol convertible bonds, assessing default risk for KRX-listed firms with cross-holding leverage (e.g., construction-sector CBs in 2024); equity-as-call gives the "intrinsic" valuation of distressed firms before bankruptcy.