

Binomial Trees

BUSS386. Futures and Options

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Lecture Outline

- One-Step Binomial Tree
- Risk-Neutral Valuation vs. DCF
- Two-Step Binomial Tree
- N-Step Binomial Tree
- Advanced Topics in Binomial Models
 - American Options
 - Determination of u and d

Why DCF Fails for Options

Standard DCF:

$$V_0 = \frac{E[CF_T]}{(1 + r_V)^T}, \quad r_V = \text{risk-adjusted discount rate.}$$

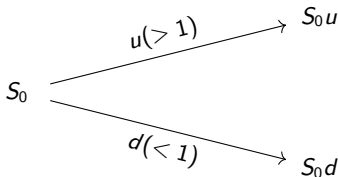
- For an option, leverage — and therefore risk — *changes with* S_t . An ATM call carries more risk than a deep-ITM call.
- No single r_V works across nodes. We'd need $r_V(S_t, K, T, \sigma)$ — itself unknown.
- Even given r_V , computing $E[S_T]$ under the real measure requires a stock model. Doubly hard.
- **No-arbitrage pricing** sidesteps both. By replicating the option with stock + bond at every node, we get a unique price *without* a discount-rate model.

Determining Option Prices

- A no-arbitrage approach, which can be implemented with binomial pricing, avoids the need to explicitly identify the relevant cost of capital
- Binomial trees incorporate the six main factors affecting the price of a stock option:
 - (1) current stock price; (2) strike price; (3) time to expiration; (4) volatility of the stock price; (5) risk-free interest rate; (6) expected dividends

Binomial Model - Setting

- Assumptions
 - Stock price follows a random walk.
: In one time step, the stock price can move up or down by a certain amount (only two possible paths).

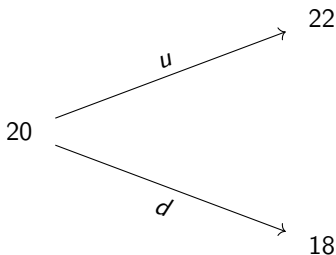


- There is no arbitrage.
- This may look too simplistic to reflect the reality. Later, we will extend the model to allow multiple steps until the option expiration.

One-Step Binomial Model

e.g. A 3-month European call option has strike price \$21. The risk-free interest rate is 12% per annum.

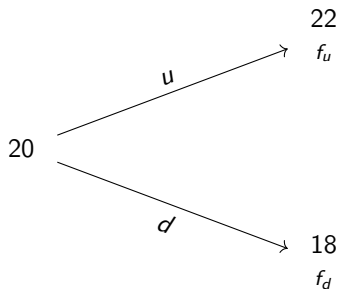
- Current stock price is \$20. The price can move either up to \$22 or down to \$18 during the life of the option.



- What is the price of the call option?

One-Step Binomial Model

- ① To price the call, we first determine option payoffs at T :



- $f_u(\text{option value when stock price is up}) = \max(22 - K, 0) = 1$
- $f_d(\text{option value when stock price is down}) = \max(18 - K, 0) = 0$

One-Step Binomial Model

- ② Next, we find a portfolio that replicates the option payoff in every case at T (using stock and bond):
- Let x denote the number of shares and y the face value of the bond (in dollar) in the replicating portfolio. We want x and y such that

$$\begin{cases} 22x + y = 1 & (\text{at } u) \\ 18x + y = 0 & (\text{at } d) \end{cases}$$

- Solving for the unknowns gives $x = 0.25$, $y = -4.5$. Thus, the replicating portfolio consists of buying 0.25 shares and selling a bond with the face value of -\$4.5.

One-Step Binomial Model

- ③ The option price at time 0 should be equal to the price of the replicating portfolio. Otherwise, an arbitrage exists.

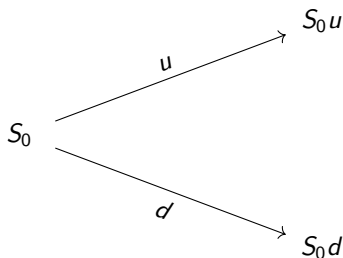
- The price of the replicating portfolio is

$$S_0x + ye^{-rT} = 20 \times 0.25 - 4.5e^{-0.12 \times 3/12} = 0.633$$

- Hence, the option price is \$0.633.
- Note that we do not consider the probabilities of going up or down!
- In fact, these probabilities are already reflected in the current stock price.

One-Step Binomial Model - General case

- A European call option has strike price K and expiration date T . The risk-free interest rate is r per annum.
- Current stock price is S_0 . The price can move either up by u or down by d during the life of the option.



- Notice that we do not assign any probability for up/down movement.

One-Step Binomial Model - General case

- 1 Find option payoffs at T :
$$\begin{cases} f_u &= \max(S_0u - K, 0) \\ f_d &= \max(S_0d - K, 0) \end{cases}$$
- 2 Find the replicating portfolio (x : number of shares, y : face value of bond).
 - We want to find x and y such that

$$\begin{cases} (S_0u)x + y = f_u & (\text{at } u) \\ (S_0d)x + y = f_d & (\text{at } d) \end{cases}$$

- Solving for the unknowns, we obtain

$$x = \frac{f_u - f_d}{S_0u - S_0d}, \quad y = \frac{uf_d - df_u}{u - d}$$

One-Step Binomial Model - General case

- ③ Calculate the present value (at time 0) of the replicating portfolio.

$$\begin{aligned} S_0x + ye^{-rT} &= \frac{f_u - f_d}{u - d} + e^{-rT} \frac{uf_d - df_u}{u - d} \\ &= e^{-rT} \left[\frac{e^{rT} f_u - e^{rT} f_d}{u - d} + \frac{uf_d - df_u}{u - d} \right] \\ &= e^{-rT} \left[\frac{e^{rT} - d}{u - d} f_u + \frac{u - e^{rT}}{u - d} f_d \right] \\ &= e^{-rT} [p \times f_u + (1 - p) \times f_d] \end{aligned}$$

where $p = \frac{e^{rT} - d}{u - d}$.

One-Step Binomial Model

e.g. Go back to the previous example of the European call option with $K = 21$, and $T = 3$ months. The risk-free interest rate is 12% per annum. The current stock price is 20. The price can either move up to 22 or down to 18 during the life of the option.

- We priced the option using the replicating portfolio. Alternatively, we can use the option pricing formula. Here, $u = 22/20 = 1.1$ and $d = 18/20 = 0.9$.

- Then,

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 1/4} - 0.9}{1.1 - 0.9} = 0.652$$

- The price of the call option is

$$\begin{aligned} e^{-rT} [pf_u + (1 - p)f_d] &= e^{-0.12 \times 1/4} [0.652 \times 1 + (1 - 0.652) \times 0] \\ &= \$0.633 \end{aligned}$$

Interpretation: Risk-Neutral Valuation

- The option price $e^{-rT} [pf_u + (1 - p)f_d]$ is similar to the form we would obtain from DCF, when p is interpreted as a probability.
- However, the form is not exactly the same as the DCF.
 - Recall that in DCF, a riskier cash flows is discounted at a higher rate, say r_{call} (e.g. CAPM).
 - However, in the result of option price, the risky option payoff is discounted at the risk-free interest rate.
- Risk-Neutral Valuation
 - The discount rate in the option price is determined **as if** investors do not require a higher return for a riskier investment, that is, as if they are **risk-neutral**.

Risk-Neutral Valuation vs. DCF

- We call p the risk-neutral probability.
- p is different from the real probability we observe in data. To distinguish, let p^* denote the real probability of an increase in the stock price.
- Risk-neutral valuation vs. DCF

	Option Pricing (Risk-Neutral Valuation)	DCF
Discount rate	r	r_{call}
Probability	p	p^*
Option price	$e^{-rT} [pf_u + (1 - p)f_d]$	$e^{-r_{\text{call}}T} [p^*f_u + (1 - p^*)f_d]$

Pricing Options Using DCF

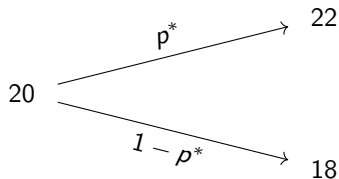
- Instead of risk-neutral valuation, we could price an option using the discounted cash flow (DCF) approach.
- To price an option using DCF, we need to find the real probability p^* and the discount rate r_{call} for the option.
- First, to find p^* , we suppose that risk-averse investors require the stock return to be α per annum in the real world.
- Then, we solve for p^* by setting α equal to the expected return in the binomial tree.

$$p^* = \frac{e^{\alpha T} - d}{u - d}$$

Why? $p^* S_0 u + (1 - p^*) S_0 d = S_0 e^{\alpha T}$

Pricing Options Using DCF

- Q1. Revisit the previous example, where stock price in 3 months is given as below. In the real world, risk-averse investors require the return on stock to be 16% per annum. What is the real probability p^* of an increase in the stock price?



Pricing Options Using DCF

- In pricing using DCF, the next step is to find the discount rate r_{call} .
- To determine the discount rate, we use the fact that a portfolio of stock and bond replicates the call option in the binomial tree.
- Then, the required return on option is the weighted average of stock return (α) and bond return (r).
- The weight is determined by the fraction of stock and bond components in the portfolio.

$$\left\{ \begin{array}{l} \text{weight on stock: } \frac{S_0 x}{S_0 x + y e^{-rT}} \\ \text{weight on bond: } \frac{y e^{-rT}}{S_0 x + y e^{-rT}} \end{array} \right.$$

Pricing Options Using DCF

Q2. In the previous tree, consider a 3-month call option with the strike price of 21. The risk-free rate is 12% per annum. What is the discount rate r_{call} for the call in DCF?

Answer: r_{call} is the weighted average of stock return and bond return, where the weight is the fraction of an asset in the entire portfolio value.

$$\begin{aligned}e^{r_{\text{call}} \times 3/12} &= \frac{S_0 x}{S_0 x + ye^{-rT}} \times e^{\alpha \times 3/12} + \frac{ye^{-rT}}{S_0 x + ye^{-rT}} \times e^{r \times 3/12} \\ &= \frac{(20)(0.25)}{0.633} \times e^{0.16/4} + \frac{-4.5e^{-0.12 \times 1/4}}{0.633} \times e^{0.12/4} \\ &= 1.112258\end{aligned}$$

Thus, the discount rate for the call is 42.56% per annum.

Pricing Options Using DCF

- Q3. Calculate the option price in Q2 using DCF. Is the price the same as the price from the risk-neutral valuation?

Risk-Neutral Valuation vs. DCF

- The option price from the risk-neutral valuation is **the same** as the price from the DCF.
- If required return on stock α is higher than the risk-free rate r , it follows that $p < p^*$.
- It implies that in the risk-neutral valuation, we amplify the probability of a bad outcome for stock investors, i.e, $1 - p$.
- We interpret this as the probability being modified to incorporate investors' risk-aversion.

Risk-Neutral Valuation

- Risk-neutral valuation is an **interpretation** of the option pricing formula obtained from the replicating portfolio.
- This does not mean that investors are risk neutral!
- We incorporate risk aversion in two ways:
 - Add risk premium to the cost of capital.
 - Increase the probability of the bad states.
- This interpretation is also useful for multi-step binomial models.

Options on Stocks Paying a Continuous Dividend Yield

- Consider a stock paying a known dividend yield at rate q . E.g. stock indices.
- The total return (**dividends** + **capital gains**) in a risk-neutral world is r .
- The dividends provide a return of q .
- Capital gains must therefore provide a return of $r - q$.
- If the stock starts at S_0 , its expected value after one time step of length Δt must be $S_0 e^{(r-q)\Delta t}$.
- This means, $pS_0u + (1 - p)S_0d = S_0 e^{(r-q)\Delta t}$.
- Hence, $p = \frac{e^{(r-q)\Delta t} - d}{u - d}$.

NB Options on currencies: $p = \frac{e^{(r-r_f)\Delta t} - d}{u - d}$.

Options on Futures

- Entering a futures contract requires **no initial investment**.
- In a **risk-neutral world**, every asset's expected return equals the risk-free rate *on invested capital*.
 - Since a futures contract requires no capital, its expected return must be zero.
- Therefore, under the risk-neutral measure:

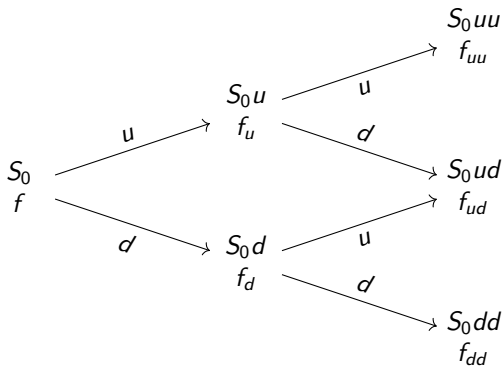
$$E^{\mathbb{Q}}[dF_t] = 0 \quad \Rightarrow \quad E^{\mathbb{Q}}[F_T] = F_0.$$

- Intuition: a zero-cost, riskless-expected-gain contract would be an arbitrage.
- This means, $pF_0u + (1 - p)F_0d = F_0$.
- Hence, $p = \frac{1-d}{u-d}$.

Two-Step Binomial Models

Two-Step Binomial Model

- The current stock price is S_0 and may go up by u or down by d in a time step. Each time step is Δt and the risk-free interest rate is r per annum.
- A European call option has the strike price of K and expires in two steps. What is the option price?



Two-Step Binomial Model

- We start at the option expiration date and find the option payoff at each stock price then.
- At $T = \Delta t$, each price and the following prices can be seen as one-step binomial tree. Thus, we can use the pricing formula of one-step models.

$$\begin{cases} f_u &= e^{-r\Delta t} [pf_{uu} + (1-p)f_{ud}] \\ f_d &= e^{-r\Delta t} [pf_{ud} + (1-p)f_{dd}] \end{cases}$$

where $p = \frac{e^{r\Delta t} - d}{u - d}$.

Two-Step Binomial Model

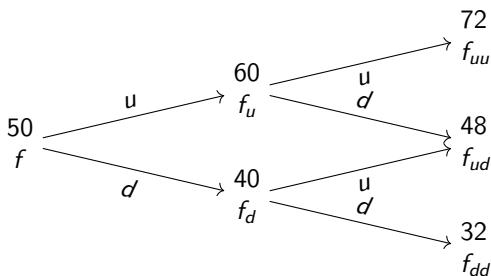
- At $T = 0$,

$$\begin{aligned}f_0 &= e^{-r\Delta t} [pf_u + (1 - p)f_d] \\&= e^{-r\Delta t} \left[p \left(e^{-r\Delta t} [pf_{uu} + (1 - p)f_{ud}] \right) + (1 - p) \left(e^{-r\Delta t} [pf_{ud} + (1 - p)f_{dd}] \right) \right] \\&= e^{-2r\Delta t} [p^2 f_{uu} + 2p(1 - p)f_{ud} + (1 - p)^2 f_{dd}]\end{aligned}$$

- This is consistent with the probabilistic interpretation of p .
 - In risk-neutral valuation, p^2 , $2p(1 - p)$, and $(1 - p)^2$ are probabilities of reaching top, middle, and bottom final nodes.

Two-Step Binomial Model - Put Option

- To price a put option, we use put payoffs at the option expiration. The rest of calculation is the same as the call valuation.
- Q. Consider a 2-year European put with $K=\$52$ on a stock with $S_0=\$50$. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 20%. The risk-free interest rate is 5%. What is the price of the put option?



N-Step Binomial Models

N-Step Binomial Model

- Suppose that there are N time steps until the option maturity and each time step is Δt .
- The risk-neutral probability of an increase in stock price during each step is $p(= \frac{e^{r\Delta t} - d}{u - d})$.
- There are $N + 1$ nodes at the expiration. Let node j denote the final stock price when the price moves upward j times and downward $N - j$ times. There, the final stock price would be

$$S_0 u^j d^{N-j},$$

where $j = 0, 1, \dots, N$.

N-Step Binomial Model

- To determine the price of an European option, we need the probability of reaching each node at the expiration.
- The probability of reaching the node j is

$$\binom{N}{j} p^j (1-p)^{N-j}$$

- There are multiple paths leading to the node j . The number of the paths is $\binom{N}{j}$, which is j -combinations from a set of N elements.
- How to calculate $\binom{N}{j}$?
 - In algebra, $\binom{N}{j} = \frac{N!}{j!(N-j)!}$.
 - In Excel, use "combin(N,j)".

N-Step Binomial Model

- For each node, the probability and option payoff is as follows:

No. of up	No. of down	Probability	Stock price at T	Option payoff
0	N	$p^0(1-p)^N$	$S_0 u^0 d^N$	f_0
1	$N-1$	$\binom{N}{1} p^1(1-p)^{N-1}$	$S_0 u^1 d^{N-1}$	f_1
\vdots	\vdots	\vdots	\vdots	\vdots
j	$N-j$	$\binom{N}{j} p^j(1-p)^{N-j}$	$S_0 u^j d^{N-j}$	f_j
\vdots	\vdots	\vdots	\vdots	\vdots
N	0	$p^N(1-p)^0$	$S_0 u^N d^0$	f_N

- The price of European option is then

$$e^{-r(N\Delta t)} \sum_{j=0}^N \binom{N}{j} p^j (1-p)^{N-j} f_j$$

where f_j is the option payoff at node j .

N-Step Binomial Model

- Q. Consider a 3-year European call with $K=\$30$ on a stock with $S_0=\$30$. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 10%. The risk-free interest rate is 5%. What is the price of the call option?

Answer: First, the option payoffs at each of 4 nodes are

$$f_0 = \max(30(1.1)^0(0.9)^3 - 30, 0) = 0$$

$$f_1 = \max(30(1.1)^1(0.9)^2 - 30, 0) = 0$$

$$f_2 = \max(30(1.1)^2(0.9)^1 - 30, 0) = 2.67$$

$$f_3 = \max(30(1.1)^3(0.9)^0 - 30, 0) = 9.93.$$

The risk-neutral probability is $p = \frac{e^{0.05 \times 1} - 0.9}{1.1 - 0.9} = 0.756$. Then, the option price is

$$e^{-0.05 \times 3} \sum_{j=0}^3 \binom{N}{j} (0.756)^j (1 - 0.756)^{N-j} f_j = 4.01$$

Pricing American Options

American Options

- In pricing American options, we should consider that the options can be exercised early.
- In a similar way to pricing European options, we build a binomial tree of stock price. Then, we start from final nodes and proceed backward.
- However, the option value at each node becomes the maximum of

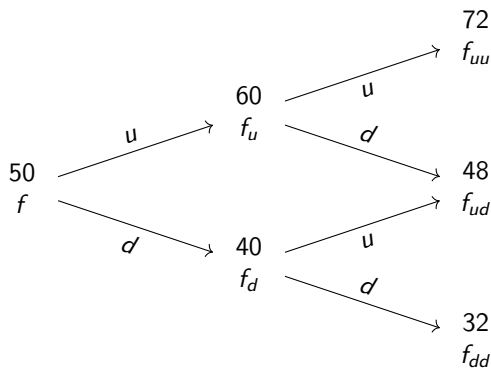
- ① the option value when delaying the exercise,

$$f = e^{-r\Delta t} [pf_u + (1 - p)f_d]$$

- ② the payoff when exercising now

American Options

e.g. Consider a 2-year American put with $K=\$52$ on a stock with $S_0=\$50$. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 20%. The risk-free interest rate is 5%. What is the price of the put option?



American Options

- The risk-neutral probability is

$$p = \frac{e^{0.05 \times 1} - 0.8}{1.2 - 0.8} = 0.6282.$$

- At final nodes, option payoffs are

$$\begin{cases} f_{uu} = \max(52 - 72, 0) = 0 \\ f_{ud} = \max(52 - 48, 0) = 4 \\ f_{dd} = \max(52 - 32, 0) = 20 \end{cases}$$

American Options

- At top node in $T = 1$, the option price is

$$f_u = \max \left(\underbrace{e^{-r\Delta t} [pf_{uu} + (1-p)f_{ud}]}_{\text{value if delay}}, \underbrace{52 - 60}_{\text{value if exercise}} \right) = \$1.415.$$

At bottom node in $T = 1$, the option price is

$$f_d = \max (e^{-r\Delta t} [pf_{ud} + (1-p)f_{dd}], 52 - 40) = \$12.$$

- At the initial node, the option price is

$$f = \max (e^{-r\Delta t} [pf_u + (1-p)f_d], 52 - 50) = \$5.090.$$

Determining u and d

Determining u and d

- We have studied how to price options when u and d are given in binomial trees.
- If they are not given, how can we determine u and d ?
- For this determination, we focus on the volatility of underlying asset.
 - The volatility σ is the standard deviation of yearly returns on the stock.
- The basic idea is to choose u and d such that the volatility in the binomial tree matches the volatility we see in data.

Determining u and d — Cox, Ross, and Rubinstein

- Once the volatility σ is obtained from data, we want to construct a binomial tree such that returns in the tree have the same volatility.
- This means that the return over one day ($= \Delta t$) should have the standard deviation of $\sigma\sqrt{\Delta t}$
 - When $\Delta t = 1/365$, $\sigma_{\Delta t} = \sigma\sqrt{1/365}$, (σ is an annual std.dev.).
- We can achieve this by choosing

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}}$$

in the binomial tree.

- Why does this choice of u and d work?

Determining u and d

- Suppose that the required return on the stock in the real world is α per annum. Then, the real probability $p^* = \frac{e^{\alpha\Delta t} - d}{u - d}$.
- Using this real probability, we can compute the variance of return $\text{Var}(r)$.
- We want to show that $\text{Var}(r)$ equals $\sigma^2\Delta t$ under this particular choice of u and d .

$$\begin{aligned}\text{Var}(r) &= E(r^2) - [E(r)]^2 \\ &= p^*(u-1)^2 + (1-p^*)(d-1)^2 - [p^*(u-1) + (1-p^*)(d-1)]^2 \\ &= \dots \\ &= (u+d)e^{\alpha\Delta t} - ud - e^{2\alpha\Delta t}\end{aligned}$$

Determining u and d

- Now, let's plug in $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$.

$$\begin{aligned}\text{Var}(r) &= (e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})e^{\alpha\Delta t} - 1 - e^{2\alpha\Delta t} \\ &= \dots?\end{aligned}$$

Determining u and d - Math Review

- Taylor series

- A function $f(x)$ can be expressed as a sum of polynomials:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

- A Taylor series of e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

- When x is small, we can ignore higher-order terms.

Determining u and d

- Applying the Taylor series to the exponential terms, we can simplify the variance.
- Here, we assume that Δt is very small, so we ignore $\Delta t^{3/2}$, Δt^2 , and higher powers.
 - For instance, $e^{\sigma\sqrt{\Delta t}} \approx 1 + \sigma\sqrt{\Delta t} + \frac{(\sigma\sqrt{\Delta t})^2}{2}$.
- Then, the variance becomes

$$\begin{aligned}\text{Var}(r) &= (e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})e^{\alpha\Delta t} - 1 - e^{2\alpha\Delta t} \\ &\approx (2 + \sigma\sqrt{\Delta t} - \sigma\sqrt{\Delta t} + \sigma^2\Delta t)(1 + \alpha\Delta t) - 1 - (1 + 2\alpha\Delta t) \\ &\approx \sigma^2\Delta t\end{aligned}$$

Estimating σ from KOSPI 200 Data

Given daily closing prices P_0, P_1, \dots, P_T , compute daily log returns:

$$r_t = \ln(P_t/P_{t-1}), \quad t = 1, \dots, T.$$

The sample standard deviation $\hat{\sigma}_{\text{day}}$ annualizes via:

$$\hat{\sigma}_{\text{annual}} = \hat{\sigma}_{\text{day}} \cdot \sqrt{252}$$

(252 trading days in a typical KRX year.)

2024 KOSPI 200 example.

- Daily log-return sd 2024 $\hat{\sigma}_{\text{day}} \approx 0.0090$.
- Annualized: $0.0090 \cdot \sqrt{252} = 0.0090 \cdot 15.87 \approx 14.3\%$.
- For a 3-month option, $\Delta t = 1/252$ per step gives $u = e^{0.143/\sqrt{252}} = e^{0.00901} = 1.00905$.

Application of Binomial Model to Corporate Finance

Application to Corporate Finance

- We have studied the binomial model as a tool for pricing options, but this model can be used to understand corporate finance.
- In particular, we can use the binomial model to find the present value of shareholders' equity and liabilities.
- The starting point is the balance sheet identity:

$$\text{Assets} = \text{Liabilities} + \text{Shareholders' Equity}$$

Application to Corporate Finance

- The market value of assets will change over time.
 - This market value is the present value of future cash flows from firms' business. As the business outlook changes time to time, the value also changes.
- Thus, we can consider a binomial tree to model future asset values.
- In this binomial tree, we can determine the present value of liabilities and equity.

Application to Corporate Finance

- Suppose that a firm has liabilities L that are due time T . Let V_T denote the firm's asset value then.

- The payoffs to creditors (debtholders) are

$$\min(V_T, L)$$

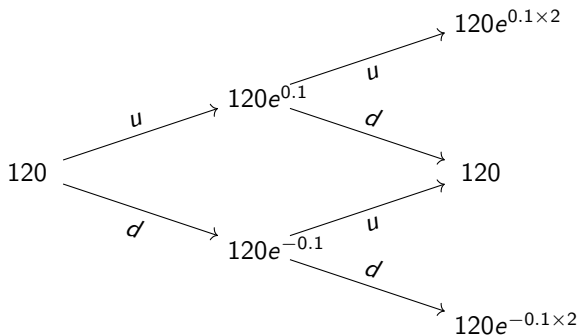
- The payoffs to shareholders (equityholders) are

$$\max(V_T - L, 0)$$

- Once the final payoffs are identified, we can calculate the present values as we do in option pricing.

Application to Corporate Finance - Example

- Q. A company's current value of assets is \$120 millions and the volatility of the asset value is 10% per annum. The company has issued a bond, so that it needs to repay \$100 millions two years later from now. The risk-free interest rate is 5% per annum. What is the current value of shareholders' equity? Use a two step binomial tree.



Application to Corporate Finance - Example

Answer: The payoffs to shareholders are

$$f_{uu} = \max(146.57 - 100, 0) = 46.57$$

$$f_{ud} = \max(120 - 100, 0) = 20$$

$$f_{dd} = \max(98.25 - 100, 0) = 0$$

The risk-neutral probability is $p = \frac{e^{0.05} - e^{-0.1}}{e^{0.1} - e^{-0.1}} = 0.731$. Then, the present value of the equity is

$$\begin{aligned} & e^{-0.05 \times 2} \left[(0.731)^2 (46.57) + 2(0.731)(1 - 0.731)(20) + (1 - 0.731)^2 (0) \right] \\ & = \$29.63 \text{ millions} \end{aligned}$$

Application to Corporate Finance - Example

Q2. Go back to the Q1. What is the present value of debt?

Summary — Binomial Pricing in One Page

- ① **One step:** replicate with Δ shares + B bond. No-arb $\Rightarrow f_0 = \Delta S_0 + B$.
- ② **RN rewrite:** $f_0 = e^{-r\Delta t}[pf_u + (1-p)f_d]$, $p = (e^{r\Delta t} - d)/(u - d)$.
- ③ **Extensions:** change drift for dividends ($r \rightarrow r - q$), futures ($r \rightarrow 0$), FX ($r \rightarrow r - r_f$).
- ④ **Multi-step:** backward induction; for American, add early-exercise check $f_{\text{node}} = \max(\text{intrinsic}, \text{hold})$ at each node.
- ⑤ **Calibrate:** $u = e^{\sigma\sqrt{\Delta t}}$, $d = 1/u$ matches volatility.
- ⑥ **Convergence:** $N \rightarrow \infty$ gives Black–Scholes (Lec 10).

Master takeaway. The binomial tree is the universal discrete-time pricing engine. Risk-neutral probability is the single concept that ties it all together.