BUSS386 Problem Set 7 — Solutions

Properties of Options: Bounds and Parity

Prof. Ji-Woong Chung

Problem 1 — Six determinants

(a)

Factor ↑	c	p
Spot S_0	↑	↓
Strike K	↓	↑
Time to expiry T	↑ (usu.)	↑ (usu.)
Volatility σ	↑	↑
Risk-free rate r	↑	↓
Dividends D	↓	↑

(b) Payoffs are kinked at K (capped at zero on the wrong side), so a wider terminal distribution adds upside without symmetric downside. Both calls and puts gain.

(c) An American put can be optimally exercised deep ITM: the interest earned on K exceeds the lost insurance value. An American call on a non-dividend stock has no such gain — holding it always dominates exercising, since you would forgo both time value and interest on K.

Problem 2 — Bounds in the strike

$(K_2 - K_1)e^{-rT} = 5e^{-0.01} = 4.9502$. Same for $(K_3 - K_2)$.

(a) Monotonicity: $7.50 \geq 5.00 \geq 2.20$ $\boxed{\text{holds}}$.

(b) Slope on $[K_1, K_2]$: $c(K_1) - c(K_2) = 2.50 \leq 4.9502$ $\boxed{\text{holds}}$.
Slope on $[K_2, K_3]$: $c(K_2) - c(K_3) = 2.80 \leq 4.9502$ $\boxed{\text{holds}}$.

(c) Convexity: $\frac{1}{2}[c(K_1) + c(K_3)] = \frac{1}{2}(7.50 + 2.20) = 4.85$ vs $c(K_2) = 5.00$.
$c(K_2) = 5.00 > 4.85 \Rightarrow$ $\boxed{\text{convexity violated}}$.
Butterfly arbitrage: short two $K_2 = 100$ calls (collect $2 \times 5.00 = 10$), long one $K_1 = 95$ call (-7.50), long one $K_3 = 105$ call (-2.20). Net premium today $= 10 - 7.50 - 2.20 = +\$0.30$. Payoff at expiry is non-negative (long butterfly always pays ≥ 0): risk-free \$0.30 today.

Problem 3 — Put-call parity check on SPX

(a) Parity (European, no dividends): $\boxed{c_0 + Ke^{-rT} = p_0 + S_0}$.

(b) LHS $= 140 + 5{,}800 \cdot e^{-0.04 \cdot 0.25} = 140 + 5{,}800 \cdot 0.99005 = 140 + 5{,}742.3 = 5{,}882.3$.
RHS $= 80 + 5{,}800 = 5{,}880$.
Discrepancy $= +\$2.30$.

(c) LHS (call + bond) is rich by \$2.30 relative to RHS (put + stock). **Sell the call, buy the put, buy the stock, short the zero**. Cash flow today: $+140 - 80 - 5{,}800 + 5{,}742.3 = \boxed{+\$2.30}$. All terminal exposures cancel (both portfolios pay $\max(S_T, K)$). Whether this is exploitable depends on whether \$2.30 exceeds bid-ask + commissions; on SPX it typically does not.

Problem 4 — Lower-bound arbitrage on a European put

(a) Lower bound: $\max(Ke^{-rT} - S_0,\ 0) = \max(40e^{-0.025} - 37,\ 0) = \max(39.012 - 37,\ 0) = \boxed{\$2.012}$. Quote $\$1.00 < \$2.012 \Rightarrow$ arbitrage.

(b) Strategy: buy the put, buy the stock, borrow Ke^{-rT}.

Action today	CF now	CF at T	
		$S_T \geq 40$	$S_T < 40$
Buy put	-1.00	0	$40 - S_T$
Buy share	-37	S_T	S_T
Borrow $40e^{-0.025}$	$+39.012$	-40	-40
Net	$\boxed{+1.012}$	$S_T - 40 \geq 0$	0

(c) Risk-free profit today $= \boxed{\$1.012}$ per share, plus a non-negative upper branch at T.

Problem 5 — Forward-style parity on KOSPI 200

$T = 0.5$, $S_0 = 360$, $K = 360$, $r = 3\%$, $q = 1.5\%$, multiplier $M =$ ₩250,000.

(a) $F_0 = S_0 e^{(r-q)T} = 360 \cdot e^{0.0075} = 360 \cdot 1.00753 = \boxed{362.71}$.

(b) Parity: $c_0 - p_0 = (F_0 - K)e^{-rT}$.
RHS $= (362.71 - 360)e^{-0.015} = 2.71 \cdot 0.9851 = \boxed{2.67}$ pts.
LHS $= 10.50 - 6.20 = 4.30$ pts.
Discrepancy $= 4.30 - 2.67 = \boxed{+1.63}$ pts. Call is rich relative to put.

(c) Sell the call, buy the put, buy e^{-qT} shares of the index, short the zero Ke^{-rT}. (Equivalently: sell the synthetic forward (long call + short put) at 4.30, buy the actual forward at 2.67.)

(d) Per contract, arbitrage $= 1.63 \cdot M = 1.63 \cdot 250{,}000 =$ [₩407,500] today (before bid-ask).

Problem 6 — Discrete-dividend parity

(a) $D = 1.50 \cdot e^{-0.04 \cdot 0.5} = 1.50 \cdot 0.9802 = \boxed{\$1.4703}$.

(b) Parity: $p_0 = c_0 + Ke^{-rT} - (S_0 - D) = 6 + 50e^{-0.04} - (52 - 1.4703) = 6 + 48.039 - 50.530 = \boxed{\$3.51}$.

(c) Doubling the dividend to \$3.00 doubles D to \$2.9405. Holding c_0 fixed and reapplying parity: $\Delta p = \Delta D = +1.4703 \Rightarrow$ new $p = 3.51 + 1.47 = \boxed{\$4.98}$ (an increase of \$1.47 per dollar of PV-dividends).

Problem 7 — Early exercise of American options

(a) **Result.** Never optimal to early-exercise an American call on a non-dividend stock; therefore $C_0 = c_0$.
Proof. For any $t < T$: $C_t \geq c_t = p_t + S_t - Ke^{-r(T-t)} = (S_t - K) + p_t + K[1 - e^{-r(T-t)}] > S_t - K$. Holding (or selling) the option dominates exercising.

(b) **Example.** $S_t = 100$, $K = 80$, $r = 4\%$ c.c., $T - t_d = 0.5$ yr remaining, large dividend $D = 5$, put-side time value $p_{t_d} \approx 0.50$. Heuristic threshold: $K(1 - e^{-r(T-t_d)}) + p_{t_d} = 80 \cdot (1 - 0.9802) + 0.50 = 1.584 + 0.50 = 2.08$. $D = 5 > 2.08 \Rightarrow$ exercise just before ex-dividend. $\boxed{\text{Optimal to exercise.}}$

(c) Exercising the put pays $K - S_t$ now; the cash K earns interest. When the put is deep ITM, the time value of waiting is small, and the interest gain on K dominates — exercising captures it.