Options II: Properties

BUSS386. Futures and Options

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Lecture Outline

- Characterizing Option Prices
  - Factors Affecting Option Prices
  - Lower and Upper Bounds for Option Prices
  - Put-Call Parity
- Reading: Chp. 11

Factors Affecting Option Prices

Question: How to Price Options?

- In this lecture, we characterize option prices as much as possible using the no-arbitrage argument.
- First, we want to know what factors matter for option prices.

Question: How to Price Options?

- From Financial Management, we learned the valuation tool, **Discounted Cash Flow**.
- According to the DCF, the option price is the present value of future option payoff.
- Consider a European call option. Then, the option price is

$$\underbrace{e^{-r_{\text{call}}T}}_{\text{discounting factor}} \times \underbrace{E\left[\max(S_T - K, 0)\right]}_{\text{option payoff}}$$

- Thus, factors affecting the option payoff or discounting will also affect the option price.

Factors Affecting Option Prices

- The following 6 factors matter for the current option price in time 0.

1. Current stock price, S_0
2. Strike price, K
3. Volatility of stock price, σ
4. Time to expiration, T
5. Risk-free interest rate, r
6. Dividends expected to be paid.

Factors Affecting Option Prices - Stock Price and Strike Price

Factor ↑	European c	European p	American C	American P
Spot S_0	↑	↓	↑	↓
Strike K	↓	↑	↓	↑

- The payoff of the call option increases with the stock price.
- The payoff of the call option decreases with the strike price K.
- The payoff of the put option behaves in the opposite way. Thus, the put price increases with K and decreases with S_0.

Factors Affecting Option Prices - Volatility

Factor ↑	European c	European p	American C	American P
Volatility σ	↑	↑	↑	↑

- Volatility is a measure of uncertainty in future stock price movement (e.g. the standard deviation).
- As volatility increases, the chance that the stock will do very well or very poorly increases.
- The owner of a call benefits from price increases but has limited downside risk in the event of price decrease.
- The owner of a put benefits from price decreases but has limited downside risk in the event of price increase.

Factors Affecting Option Prices - Future Dividends

Factor $\uparrow$	European c	European p	American C	American P
Dividends D	$\downarrow$	$\uparrow$	$\downarrow$	$\uparrow$

- When the current stock price is fixed, more future dividends until the option expiration lowers the future stock price on the expiration date.
- The call price decreases with the expected amount of future dividends.
- The put price increases with the expected amount of future dividends.

Factors Affecting Option Prices - Time to Expiration

Factor ↑	European c	European p	American C	American P
Time T	?	?	↑	↑

- For American options, options with a longer time to expiration are worth more or as much as options with a shorter time to expiration.
- This is because that the owner of a long-life option has all exercise opportunities open to the owner of a short-life option and more.
- For European options, generally, they are more valuable with a longer time to expiration.
  - However, there are some exceptions, for example, when a large dividend is expected to be paid within the time to expiration or when a company is likely to default.

Factors Affecting Option Prices - Risk-Free Rate

Factor ↑	European c	European p	American C	American P
Risk-free rate r	↑	↓	↑	↓

- An increase in interest rate will decrease the PV of strike price.
- For call options, the expected payoff is $E[S_T] - PV(K)$. Hence, the option becomes more valuable with a higher interest rate.
- For put options, the expected payoff is $PV(K) - E[S_T]$. Hence, the option becomes less valuable with a higher interest rate.

Factors Affecting Option Prices

Q. Consider two European put options with the same strike prices and the same expiration dates but for different underlying stocks A and B. Stock A has the volatility of 20% and is expected to pay dividend $3 in a year. The stock B has the volatility of 15% and is expected to pay dividend $2 in a year. The current stock prices of A and B are the same. Can we determine which put option has the higher price?

Lower/Upper Bounds for Option Prices

Put-Call Parity

Properties of Stock Options

- To exactly determine the option price, we need a model that describes futures stock price.
- We don't want to use a model yet.
- Instead, we use the no-arbitrage argument only and try to characterize the option price as much as possible.
- This results in ...
  - Lower/upper bound for the option price
  - Put-call parity

European vs American Options

- Consider an American option and a European option with the same strike prices, expiration dates, and underlying assets.

- The American option is always more valuable or as valuable as the European option.

$$c_0 \leq C_0 \quad \text{and} \quad p_0 \leq P_0$$

- This is because the owner of the American option has all exercise opportunities open to the owner of the European option and more.

- This relationships hold for all types of options irrespective of whether the underlying asset pays dividends or not.

Properties of Stock Options

- We consider lower/upper bounds and the put-call parity case by case.

1. Non-dividend-paying stock
    - European options
    - American options
2. Dividend-paying stock
    - Continuous dividend
        - European options
        - American options
    - Discrete dividend
        - European options
        - American options

Lower/Upper Bounds for Option Prices

Put-Call Parity

I. Options on Non-Dividend-Paying Stocks

Upper Bounds — Non-Dividend Stock

Calls. The option to buy S at K is never worth more than S itself (buying the stock outright is at least as good):

$$c_0 \;\le\; S_0, \qquad C_0 \;\le\; S_0.$$

Puts. European put pays at most K at T. Discounting:

$$p_0 \;\le\; K\,e^{-rT}.$$

American put can be exercised any time, so its discounted max payoff is just K:

$$P_0 \;\le\; K.$$

- These are crude but *always* hold; useful sanity checks on quotes.
- During LTCM week (Aug–Sep 1998) some illiquid OTC put quotes *exceeded* K briefly — a sign that the dealer was compensating for credit risk, not pure option value.

Lower Bound — European Call (Non-Dividend)

$$c_0 \;\geq\; \max\bigl(S_0 - K\,e^{-rT},\;0\bigr).$$

Proof sketch. Compare two portfolios:

Portfolio	Today	At T
(A) Long call + bond paying K at T	$c_0 + Ke^{-rT}$	$\max(S_T, K)$
(B) Long one share	S_0	S_T

At T, A's payoff $\geq$ B's payoff (since $\max(S_T, K) \geq S_T$). By no-arbitrage:

$$c_0 + Ke^{-rT} \;\geq\; S_0 \;\Longrightarrow\; c_0 \;\geq\; S_0 - Ke^{-rT}.$$

Combining with $c_0 \geq 0$ gives the boxed bound.

Example — Call Lower-Bound Arbitrage

Q. $S_0 = \$20$, $K = \$18$, $r = 10\%$, $T = 1$. A call is quoted at \$3. Arbitrage?

- **A.** Lower bound $= \max(20 - 18e^{-0.1}, 0) = \max(20 - 16.287, 0) = \boxed{\$3.713}$.
- Call price $\$3 < \$3.713 \Rightarrow$ **arbitrage**: buy the call, short the stock, lend Ke^{-rT}:

Action today	CF now	CF at T: $S_T \geq 18$	CF at T: $S_T < 18$
Buy call	-3	$S_T - 18$	0
Lend $18e^{-0.1}$	-16.287	18	18
Short share	$+20$	$-S_T$	$-S_T$
Net	$\boxed{+0.713}$	0	$18 - S_T \geq 0$

Positive cash today, non-negative cash at T. "Cashflow arbitrage" — the lower branch is a free option, not a wash.

Lower Bound — European Put (Non-Dividend)

$$p_0 \;\ge\; \max\!\big(K\,e^{-rT} - S_0,\; 0\big).$$

Proof sketch. Compare:

Portfolio	Today	At T
(C) Long put + one share	$p_0 + S_0$	$\max(S_T, K)$
(D) Long bond paying K at T	Ke^{-rT}	K

At T, C $\ge$ D (since $\max(S_T, K) \ge K$):

$$p_0 + S_0 \;\ge\; Ke^{-rT} \;\Longrightarrow\; p_0 \;\ge\; Ke^{-rT} - S_0.$$

Q. $K = \$40,\; T = 0.5,\; S_0 = \$37,\; r = 5\%,\; p_0 = \$1.00$. Arbitrage?
A. Lower bound $= 40e^{-0.025} - 37 = 39.012 - 37 = \boxed{\$2.012}$. Quote $\$1 < \$2.012$ $\Rightarrow$ buy the put, buy the stock, borrow Ke^{-rT}.

Put-Call Parity - Derivation 1

[European Call and Put]

$$\boxed{c_0 + K\,e^{-rT} \;=\; p_0 + S_0}$$

- Consider the previous portfolios:
  1. European call + bond that will pay K at T
  3. European put + one share

- At the option expiration T, ① and ③ always generate the same cash flows:
  1. $\max(S_T - K, 0) + K = \max(S_T, K)$
  3. $\max(K - S_T, 0) + S_T = \max(K, S_T)$

- Hence, under no-arbitrage, current prices should satisfy

$$c_0 + Ke^{-rT} = p_0 + S_0$$

Put-Call Parity - Derivation 2

[European Call and Put]

$$\boxed{c_0 - p_0 \; = \; (F_0 - K)\, e^{-rT}}$$

- Consider a portfolio of options: + long Call + short Put
- The payoff of this portfolio at T is $S_T - K$, which is the same as the payoff of a forward with $F_0 = K$.
- The present value of the payoff is $S_0 - Ke^{rT}$.
- Therefore, $c_0 - p_0 = S_0 - Ke^{-rT}$.

Example — Arbitrage with Parity

[European Call and Put]

Q. Suppose that $S_0 = \$31$, $K = \$30$, $r = 10\%$, $T = 3$ month. The price of a European call is \$3 and the price of a European put is \$2.25. Is there an arbitrage opportunity? If so, find an arbitrage strategy and its profit.

Answer: Let's check whether the put-call parity holds:

1. $c_0 + Ke^{-rT} = 3 + 30e^{-0.1\times 3/12}$=32.26
3. $p_0 + S_0 = 2.25 + 31 = 33.25$

So, the portfolio ③ is overpriced relative to portfolio ①. Then, the arbitrage strategy is

Action today	Today	T	
		$S_T \geq 30$	$S_T < 30$
long call	-3	$(S_T - 30)$	0
buy a bond	-30$e^{-0.1\times 3/12}$	30	30
short put	2.25	0	$-(30 - S_T)$
sell share	31	$-S_T$	$-S_T$
net	0.99	0	0

American Options - Early Exercise

[Non-dividend-paying stock]

- Long position in an American option has the right to exercise earlier than the expiration.
- In a special case, when we long an **American call on a non-dividend-paying stock**, it is **never optimal to exercise early** before the expiration (for reason we will see below).
- Why is early exercise not optimal for this special case?
  - At time t, the option holder has the value

$$C_t = \max \begin{cases} \text{Value of waiting} & \text{if not exercise} \\ S_t - K & \text{if exercise} \end{cases}$$

American Options - Early Exercise

[Non-dividend-paying stock]

- If $C_t > S_t - K$, we can conclude that early exercise is not optimal.
- It turns out that the above inequality holds at any $0 \leq t < T$.
  - As an American call option is worth more or as much as a European call,
  $$C_t \geq c_t$$
  - From the put-call parity of European options, we have
  $$\begin{aligned} c_t &= p_t + S_t - Ke^{-r(T-t)} \\ &= p_t + (S_t - K) + K(1 - e^{-r(T-t)}). \end{aligned}$$
  Thus, $c_t > S_t - K$.
  - Hence, $C_t > S_t - K$. It's better to sell the call.

American Options - Early Exercise

[Non-dividend-paying stock]

- Alternatively ...

- Consider the following two portfolios:
  1. American call
  2. One share + short bond that will pay K at T

- At the option expiration T, ① always generates larger cash flows than ②:
  1. $\max(S_T - K, 0)$
  2. $S_T - K$

- Hence, under no-arbitrage, at $t < T$,

$$C_t \geq S_t - Ke^{-r(T-t)} \geq S_t - K$$

American Options - Early Exercise

[Non-dividend-paying stock]

- In the case of an American **put** option, sometimes it is optimal to exercise early.
  - $P_t \leq K$ because the maximum payoff from the option is $K - 0$.
  - Suppose $S_t = 0$, the payoff at t is K and we know $P_t \leq K$.
  - It's better to receive K earlier than later.
  - Hence, it can be desirable to exercise the option at $t \leq T$.

Lower Bounds for American Call

[Non-dividend-paying stock]

- We just proved that the option expiration is the only date that we may exercise an American call on non-dividend-paying stock.
- This means that the European and the American calls will deliver the same cash flows. Thus, $C_0 = c_0$.
- Hence, the lower bound of American call is the same as the lower bound of European call.

Lower Bounds for American Put

[Non-dividend-paying stock]

- For American put option, it is sometimes optimal to exercise early, in particular, when the option is deep in the money.

- At time t, the option holder has the value

$$P_t = \max \begin{cases} \text{Value of waiting} & \text{if not exercise} \\ K - S_t & \text{if exercise} \end{cases},$$

  Thus, we know $P_t \geq K - S_t$.

- Combining the fact that the put option price cannot be negative, the lower bound becomes

$$P_0 \geq \max(K - S_0, 0).$$

Put-Call Parity for American Options

- For American options on non-dividend-paying stocks, the put-call parity is

$$S_0 - K \leq C_0 - P_0 \leq S_0 - Ke^{-rT}$$

- Let's prove the right inequality first and then prove the left inequality.

Put-Call Parity for American Options - Right Inequality

- As $P_0 \geq p_0$, it follows that $C_0 - P_0 \leq C_0 - p_0$. Also, we know that $C_0 = c_0$ for non-dividend-paying stock. Thus,

  $$C_0 - P_0 \leq C_0 - p_0 = c_0 - p_0$$

  From the put-call parity for European options, we know that $c_0 - p_0 = S_0 - Ke^{-rT}$. Thus,

  $$C_0 - P_0 \leq c_0 - p_0 = S_0 - Ke^{-rT},$$

  which proves the right inequality.

Put-Call Parity for American Options - Left Inequality

- To prove the left inequality, we consider the following two portfolios:

portfolio A:	American call + bond worth K now
portfolio B:	American put + stock

- We want to prove that the value of the portfolio A is higher than or equal to the value of portfolio B. This will lead to the left inequality, $C_0 - P_0 \geq S_0 - K$.

- In derivation, we consider the two different cases:

  1. case 1: put option is exercised earlier than the expiration.
  2. case 2: put option is not early-exercised.

Put-Call Parity for American Options - Left Inequality - Case 1

- Suppose that the put option is exercised earlier than the expiration, say t $(0 \leq t < T)$.

- Then, the portfolio value at time t is

$$\begin{aligned} &\text{portfolio A:} && C_t + Ke^{rt} \\ &\text{portfolio B:} && (K - S_t) + S_t \end{aligned}$$

- As $C_t \geq 0$ and $e^{rt} \geq 1$, we can conclude that the time-t value of portfolio A is higher than or equal to the value of portfolio B.

Put-Call Parity for American Options - Left Inequality - Case 2

- Suppose that the put option is NOT exercised earlier than the expiration. Then, the put option may or may not be exercised on the expiration T.

- If $S_T \geq K$ on the expiration T,

$$\begin{array}{ll} \text{portfolio A:} & (S_T - K) + Ke^{rT} \\ \text{portfolio B:} & 0 + S_T \end{array}$$

  Thus, the portfolio A has the higher value.

- If $S_T < K$ on the expiration T,

$$\begin{array}{ll} \text{portfolio A:} & 0 + Ke^{rT} \\ \text{portfolio B:} & K - S_T + S_T \end{array}$$

  Again, the portfolio A has the higher value.

Put-Call Parity for American Options - Left Inequality - Case 2

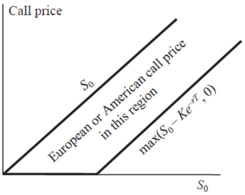
- Thus, we can conclude that the time-T value of portfolio A is higher than the value of portfolio B.
- $C_0 + K \geq P_0 + S_0 \Longrightarrow C_0 - P_0 \geq S_0 - K$.

Summary of Bounds for Option Prices

[Non-dividend-paying stock]

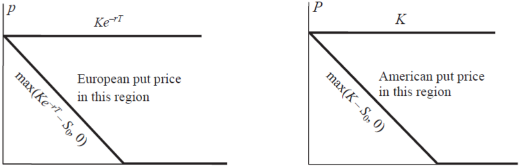
Call on non-dividend-paying stock

Figure 10.3 Bounds for European and American call options when there are no dividends.



Put on non-dividend-paying stock

Figure 10.5 Bounds for European and American put options when there are no dividends.



Lower/Upper Bounds for Option Prices
Put-Call Parity

II. Options on Dividend-Paying Stocks

Properties of Options on Dividend-Paying Stock

- Recall that the underlying asset's dividend payment affects option prices.
- Hence, the bounds and put-call parity should be modified for options on dividend-paying stock.
- For European options on dividend-paying stocks, we can find the result via a shortcut:
    - Starting from the results for non-dividend-paying stocks, we replace S_0 with the ex-dividend component.

Lower/Upper Bounds and Put-Call Parity for European Options - Discrete Dividends

- Suppose that the underlying assets pay discrete dividends. Let D denote the present value of futures dividends until the option expiration.

- For European call,

$$\max\left(S_0 - D - Ke^{-rT}, 0\right) \leq c_0 \leq S_0 - D$$

- For European put,

$$\max\left(Ke^{-rT} - (S_0 - D), 0\right) \leq p_0 \leq Ke^{-rT}$$

- Put-Call parity,

$$c_0 + Ke^{-rT} = p_0 + S_0 - D$$

Lower/Upper Bounds and Put-Call Parity for European Options - Discrete Dividends

Q. A European call option on a stock with $K = 20$ and $T = 3$ is priced at \$9. The current stock price is \$30, and the stock is expected to pay dividend of \$2 in $T = 1$ and $T = 2$. The risk-free interest rate is 3%. What is the price of a European put option with the same strike price and expiration date?

Lower/Upper Bounds and Put-Call Parity for European Options - Continuous Dividends

- Suppose that the underlying assets pay continuous dividends with the dividend yield q per annum.

- For European call,

$$\max\left(S_0 e^{-qT} - Ke^{-rT}, 0\right) \leq c_0 \leq S_0 e^{-qT}$$

- For European put,

$$\max\left(Ke^{-rT} - S_0 e^{-qT}, 0\right) \leq p_0 \leq Ke^{-rT}$$

- Put-Call parity,

$$c_0 + Ke^{-rT} = p_0 + S_0 e^{-qT}$$

Summary — Bounds and Parity at a Glance

	Call lower / upper	Put lower / upper
Eur, no div	$\max(S_0 - Ke^{-rT}, 0)$ / S_0	$\max(Ke^{-rT} - S_0, 0)$ / Ke^{-rT}
Eur, disc div	$\max(S_0 - D - Ke^{-rT}, 0)$ / $S_0 - D$	$\max(Ke^{-rT} - (S_0 - D), 0)$ / Ke^{-rT}
Eur, cont div	$\max(S_0 e^{-qT} - Ke^{-rT}, 0)$ / $S_0 e^{-qT}$	$\max(Ke^{-rT} - S_0 e^{-qT}, 0)$ / Ke^{-rT}
Am, no div	= Eur call	lower = $\max(K - S_0, 0)$; upper = K

One-line parity. $c_0 - p_0 = (F_0 - K)\, e^{-rT}$

Takeaways.

1. Bounds and parity are *model-free*; they hold whenever the market is frictionless.
2. When they break, suspect frictions: bid-ask, credit risk, short-sale constraints, illiquidity.