Pricing Forwards and Futures

BUSS386. Futures and Options

Prof. Ji-Woong Chung

Lecture Outline

- No Arbitrage Argument
- Determination of Forward Prices
- Valuing Forward Contracts
- Comparison Between Forward and Futures Prices
- Reading: Ch. 5

Forward pricing — why we care

- Last week we *used* the forward price F_0 as a market quote and built hedges around it.
- Today we *derive* F_0 from first principles — spot price, the risk-free rate, and any carry costs (dividends, storage, convenience yield).
- One simple no-arbitrage idea (cost of carry) generates the entire family of pricing formulas:

$$F_0 = S_0 \, e^{(r-q+u-y)\,T}.$$

- Same machinery works for stocks, bonds, currencies, and commodities — just plug in the right q, u, y.

No Arbitrage Argument

Arbitrage

- **Arbitrage** is a trade where investors can make “free lunch” profits.
- For instance, if we see a price difference for the same assets, we can make an arbitrage profit (buy low and sell high).

e.g. Suppose that a stock is traded in both New York Stock Exchange and London Stock Exchange. Its price in New York is \$140, while it is £100 in London. The exchange rate is \$1.43 per pound.

  - Buy a share in New York and sell it in London.
  - Profit = 100×1.43 - 140 = \$3. This profit is risk-free.

Arbitrage - Definition

- Formally, we claim that a trading strategy is an arbitrage if it satisfies the following conditions.

  1. It always generates **non-negative** cash flows, and
  2. It sometimes generates **positive** cash flows.

No Arbitrage Argument

- In the markets, there are numerous investors looking for any arbitrage opportunity.
- Suppose that an arbitrage exists for a certain asset.
- Due to forces of supply and demand, the prices will eventually change. In equilibrium, the prices of one asset will be the same across different markets.
- Generally, arbitrage opportunities quickly disappear.

No Arbitrage Argument

- We can apply the no arbitrage argument to two assets (portfolios) A and B that will generate the same cash flows in the future in every condition.

- The current prices of assets A and B should be the same. Otherwise, an arbitrage exists.

- If current prices are different, we can make an arbitrage through "**buy low and sell high**".

  $\Rightarrow$ However, an arbitrage should NOT exist in a purely competitive financial market.

Assumptions: Short Selling

- In constructing an arbitrage, we assume that the market allows short selling.

Def. Short selling is selling an asset that we do not own.

e.g. Suppose that an investor wants to short a stock at time 0 at the current price of \$120.

  - At time 0, the investor borrows the stock, sells immediately, and receives the proceeds of \$120.
  - One year later, stock price falls to \$100. To close the position, the investor buys the stock and return it back to the original owner.

Action	year 0	year 1
(Short) Sell a stock	120	-100

Assumptions: Short Selling

- What if the share pays dividend?
- Then, the shorting investor needs to pay the dividend to the original owner.

e.g. An investor shorts a stock at time 0 whose current price is \$120. The stock pays \$5 dividend in six month.

- Again, by borrowing and selling immediately, the investor receives \$120.
- In six month, the investor provides the original owner with the \$5 dividend.
- One year later, stock price falls to \$90. To close the position, the investor buys the stock and return it back to the original owner.

Action	year 0	year 0.5	year 1
(Short) Sell a share	120	-5	-90

Determination of Forward Prices

Determination of Forward Prices - Basic Idea

- Investors enter a long or short position in forward contract **at zero cost**.
- In other words, the value of forward contract should be **zero** at the time of initiating the contract.
- Conversely, we can determine the forward price, so that the current value of forward contract becomes zero.

Determination of Forward Prices

- Consider an underlying asset that pays no dividends. Its current price is S_0.

- What should be the forward price?

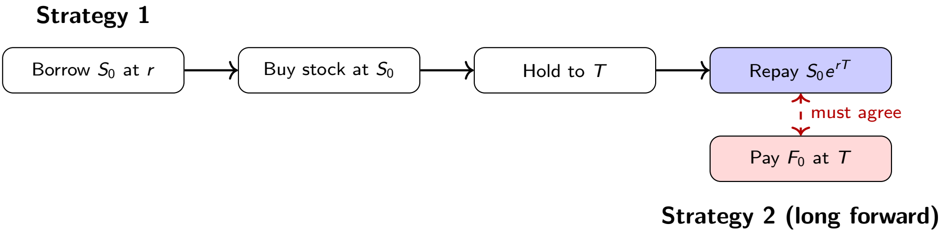
The cost-of-carry argument

Two ways to own one share at time T:

1. **Buy spot, carry to T.** Borrow S_0 at rate r, buy the stock, hold. Net cost at T: $S_0\, e^{rT}$.
2. **Long the forward.** Pay nothing today; agree to pay F_0 at T.

Both deliver one share at T. By no arbitrage, the costs at T must agree:

$$\boxed{F_0 \;=\; S_0\, e^{rT}.}$$



Determination of Forward Prices - Arbitrage

- What if

$$F_0 \neq S_0 e^{rT}?$$

$\Rightarrow$ An arbitrage exists.

e.g. Consider a 3-month forward contract on a stock whose current price is \$40. The 3-month risk-free interest rate is 5% per annum.

1. What if the forward price is 43 ($> 40e^{0.05\times 3/12}$)?

   $\Rightarrow$ There is an arbitrage:

Action	Cash flow in 0	Cash flow in 3 month
buy stock	-40	S_T
short forward	0	$43 - S_T$
sell bond	40	$-40e^{0.05\times 3/12}$
net	0	2.497

Determination of Forward Prices - Arbitrage

2. What if the forward price is 39 ($< 40e^{0.05\times 3/12}$)?

   $\Rightarrow$ There is another arbitrage strategy:

Action	Cash flow in 0	Cash flow in 3 month
sell stock (short selling)	40	$-S_T$
buy forward	0	$S_T - 39$
buy bond	-40	$40e^{0.05\times 3/12}$
net	0	1.503

Determination of Forward Prices - Example

Q. Consider a 1-year forward contract on a stock whose current price is \$50. The forward price is \$51, and the risk-free interest rate is 7% per annum. Is there an arbitrage? If so, show the arbitrage strategy.

Action	Cash flow in 0	Cash flow in 1 year
net		

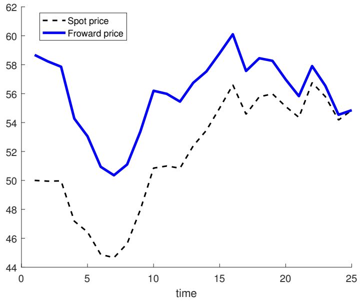
Forward and Spot Prices

- Consider a forward contract initiating at time t. Given the maturity date T, the forward price is

$$F_t = S_t e^{r(T-t)}$$

- Thus, the forward and spot prices are usually different. Only at the expiration, they become the same.

- Also, the forward price changes through time.



Determination of Forward Prices for Underlying Assets Paying Dividends

Dividend Payment and Forward Prices

- Until now, we have assumed that the underlying assets in forward do not pay any dividends.
- What if the underlying asset will pay dividends in the future? Are there changes in forward prices?

$\Rightarrow$ Yes, because...

- The current price S_0 of the underlying asset includes the future dividends.
- However, a long/short position in forward will not receive the dividends. Also, the forward payoff is determined by the ex-dividend price.

Determination of Forward Prices - Discrete Dividends

- We consider two different forms of dividend payments.
  1. Discrete dividends: dividends will be paid at certain points in time.
  2. Continuous dividends: dividends will be paid at every instant continuously.

- We first consider the case of discrete dividends.

- Suppose that stock pays dividends until the maturity T. The present value of all future dividends is I.

- The forward price is

$$F_0 = (S_0 - I)e^{rT}$$

Determination of Forward Prices - Discrete Dividends

- Why? Consider the following two portfolios:
  1. long forward with F_0 + buy a bond that will pay $F_0 + Ie^{rT}$ at T
  2. buy a stock

- At the contract maturity T, the two portfolios have the same cash flows:
  1. $(S_T - F_0) + F_0 + Ie^{rT}$
  2. $(S_T + Ie^{rT})$

- The portfolio values are the same at T. Thus, their current values are the same:

$$0 + F_0 e^{-rT} + I = S_0$$

Discrete dividends — worked example

Q1. A 9-month forward on a corporate bond. Spot bond price \$900; \$40 coupon in 4 months; 4-month rate 3%, 9-month rate 4% (both c.c.). Find the no-arbitrage forward.

A1. PV of coupon: $I = 40\,e^{-0.03\times 4/12} = 39.60$.

$$F_0 = (900 - 39.60)\,e^{0.04\times 9/12} = \boxed{\$886.60}.$$

Q2. Suppose the market quote is $F_0^{\text{mkt}} = \$910$. Arbitrage?

A2. $\$886.60 < \$910 \Rightarrow$ market forward too high. Short the forward, buy the bond, finance with two zeros:

Action	CF at $t = 0$	CF at 4m	CF at 9m
Buy corporate bond	-900	$+40$	S_T
Short forward	0	0	$910 - S_T$
Sell 4m zero (notional 40)	$40\,e^{-0.03\times 4/12} = 39.60$	-40	0
Sell 9m zero (notional 910)	$910\,e^{-0.04\times 9/12} = 883.11$	0	-910
Net	$\boxed{+\$22.71}$	0	0

Determination of Forward Price - Continuous Dividends

- Some securities pay continuous dividends (e.g, stock index, foreign currency).
  - Once we invest in a stock index, dividends from each individual stock will be paid at different points of time.
  - Having a lot of stocks in the index, we can approximate the index as paying dividends continuously.
- To simplify the argument, we assume that the dividends will be reinvested immediately to buy more shares.

Determination of Forward Price - Continuous Dividends

- Let q denote the dividend yield per annum. Stock price at time 0 is S_0.
  - Let N denote the number of dividend payments in a year.
  - In one period, investor receives dividend $\frac{q}{N}S_t$.
  - Reinvesting the dividend, the investor owns $\frac{q}{N}$ additional shares. Thus, the number of shares increases by factor of $\left(1+\frac{q}{N}\right)$ in one period.
  - When investing for one year, the number of shares increases by factor of $\left(1+\frac{q}{N}\right)^N$. If N becomes infinitely large, the factor becomes e^q.
- If we invest for T years, the number of shares increases by e^{qT}.

Determination of Forward Price - Continuous Dividends

- What if the underlying asset pays continuous dividends with dividend yield q per annum?

- Forward price is

$$F_0 = S_0 e^{(r-q)T}$$

- Why? Consider the two portfolios:
  1. long forward with F_0 + buy a bond that will pay F_0 at T
  2. buy e^{-qT} share of stock

- The two portfolios will have the same cash flows at T:
  1. $(S_T - F_0) + F_0$
  2. $S_T e^{-qT} e^{qT}$

- Therefore, the two portfolios should have the same present values:

$$0 + F_0 e^{-rT} = S_0 e^{-qT}$$

Determination of Forward Price - Continuous Dividends - Foreign Currency

- If we hold a foreign currency, we receive interests that are paid continuously at the risk-free rate prevailing in the foreign country.

- Thus, foreign currency can be regarded as an asset with continuous dividends.

- Forward price is then

$$F_0 = S_0 e^{(r - r_f)T}$$

  where r_f is the foreign risk-free rate.

FX forward — USD/KRW worked example

Setup. 1-year USD/KRW forward. Today: spot $S_0 = 1{,}380$ per USD; KRW rate (BoK base, c.c.) $r = 2.5\%$; USD rate (Fed funds, c.c.) $r_f = 4.0\%$.

Q1. Find the no-arbitrage 1-year forward.

A1. $F_0^* = 1{,}380 \cdot e^{(0.025-0.040)\times 1} = 1{,}380 \cdot 0.9851 = \boxed{1{,}359.4}$ per USD.

Note: $r < r_f$ (KRW pays less than USD), so $F_0 < S_0$ — USD is at a forward *discount* in KRW. This is *covered interest parity*.

Q2. Market quote is $F_0^{\text{mkt}} = 1{,}380$ (forward equals spot). Arbitrage?

A2. $F_0^{\text{mkt}} > F_0^* \Rightarrow$ forward too *high*. Sell USD forward at the high market price:

Action (Korean investor)	CF today (KRW)	CF in 1 yr (KRW)
Buy $e^{-0.04} = 0.9608$ USD spot at ₩1,380	$-1{,}325.9$	USD deposit grows to \$1 $\to S_T$
Short USD/KRW forward (1 USD at ₩1,380)	0	$1{,}380 - S_T$
Borrow ₩1,325.9 at 2.5%	$+1{,}325.9$	$-1{,}359.5$
Net	0	$\boxed{+₩20.5}$

Risk-free profit per USD $\Rightarrow$ market quote must converge to F_0^*.

Determination of Forward Price - Commodities

- Storing commodities has costs and benefits.
- Forward price with proportional storage cost u

$$F_0 = S_0 e^{(r+u)T}$$

- Forward price with convenience yield y

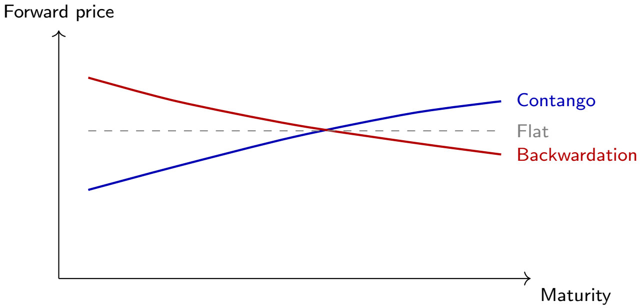
$$F_0 = S_0 e^{(r-y)T}$$

- Together

$$F_0 = S_0 e^{(r-y+u)T}$$

The shape of the forward curve

- Contango is a pattern of forward prices that increases with contract maturity
- Backwardation is a pattern of forward prices over time that decreases with contract maturity



Commodities that cannot be stored

- May be no storage or very limited storage life: electricity, lettuce, strawberries, temperature, rainfall, ...
- For non-storable commodities, forward prices can have information about future spot prices because no-arbitrage conditions don't hold (i.e. not tied to current price)
  - Approach to pricing is to model stochastic future spot prices
  - Also must infer discount rates

Summary

- For stocks, bonds, currencies, metals, or storable commodities, the forward price does not reveal any additional information beyond what is already reflected in the spot price.
- The forward price is determined entirely by no-arbitrage conditions—it depends only on the current spot price, the risk-free rate, and cash flows (dividends, coupons, storage costs, or convenience yield) between today and maturity T.
- The forward price should not be used as a forecast of the future spot price—it reflects pricing under risk-neutral valuation (risk-free rate), not market expectations.
  - For a non-dividend-paying stock: $E[S_T] = S_0 e^{\alpha T}$, where $\alpha =$ expected return. $F = S_0 e^{rT}$. Since $\alpha > r$, the expected future spot price exceeds the forward price.

Valuing Forward Contracts

Valuing a live forward contract

- At inception, value $f = 0$. Over time, the forward price moves and the contract becomes worth something.
- Suppose at $t > 0$ the new forward (for delivery at the same T) is quoted at F_t.
- **Replication:** long forward at F_0 + lend $(F_0 - F_t)\, e^{-r(T-t)}$ today $\equiv$ long forward at F_t (cost 0).
- Equating values today:

$$\boxed{f_{\text{long}} = (F_t - F_0)\, e^{-r(T-t)}, \qquad f_{\text{short}} = (F_0 - F_t)\, e^{-r(T-t)}.}$$

- In words: as the new forward price drifts away from your locked F_0, you accumulate value (or loss) equal to the discounted spread.

Valuing a live forward — written in S_t

Fold the cost-of-carry $F_t = S_t e^{r(T-t)}$ (etc.) into the valuation:

$$f_{\text{long}} = \begin{cases} S_t - F_0\, e^{-r(T-t)} & \text{no dividend} \\ (S_t - I) - F_0\, e^{-r(T-t)} & \text{discrete dividends} \\ S_t\, e^{-q(T-t)} - F_0\, e^{-r(T-t)} & \text{continuous dividends} \end{cases}$$

Same intuition: today's replicating-cost minus the locked obligation, both pulled to time t.

Valuing a live forward — worked example

Q. An investor entered a 1-year long forward on a non-dividend stock when the spot was \$40 and $r = 5\%$ (c.c.). Two months later, the spot has risen to \$45. What is the forward worth?

A. Original forward price: $F_0 = 40\,e^{0.05} = 42.05$.
New forward price (10 months left): $F_t = 45\,e^{0.05\times 10/12} = 46.91$.
Value of the long position at $t = 2$ months:

$$f_{\text{long}} = (F_t - F_0)\,e^{-r(T-t)} = (46.91 - 42.05)\,e^{-0.05\times 10/12} = \boxed{\$4.66}.$$

Equivalently in S_t: $f_{\text{long}} = S_t - F_0\,e^{-r(T-t)} = 45 - 42.05 \cdot 0.9592 = \$4.66$. ✓

Forward vs. Futures Prices

Forward vs. futures prices — the cash-flow gap

- Same underlying, same delivery T. Both deliver $\sim S_T$ in expectation.
- Forward: one cash flow at T ($S_T - F_0$ for long). Futures: a stream of daily marks $F_t - F_{t-1}$, summing to $S_T - F_0$ at T.

Day	Forward CF	Futures CF
0	0	0
1	0	$F_1 - F_0$
2	0	$F_2 - F_1$
$\vdots$	$\vdots$	$\vdots$
T	$S_T - F_0$	$S_T - F_{T-1}$

- If $r = 0$, cumulative futures gain = forward payoff $\Rightarrow$ *forward price* = *futures price*.
- If r is constant or known, the same equality holds (Cox–Ingersoll–Ross 1981).

Forward vs. futures — when the rate is stochastic

When r moves and is correlated with the spot, daily futures gains earn (or pay) interest on the way to T. The sign of the wedge depends on the correlation:

$\rho(\Delta r, \Delta S) > 0$

- $S \uparrow \Rightarrow$ futures gain, reinvested at higher r (good).
- $S \downarrow \Rightarrow$ futures loss, financed at lower r (good).
- $\Rightarrow$ **Futures price $>$ Forward price.**

$\rho(\Delta r, \Delta S) < 0$

- $S \uparrow \Rightarrow$ futures gain, but reinvested at *lower* r.
- $S \downarrow \Rightarrow$ futures loss, financed at *higher* r.
- $\Rightarrow$ **Futures price $<$ Forward price.**

Forward vs. futures — in practice

- For most underlyings (equities, FX, most commodities), $\text{Cov}(\Delta r, \Delta S)$ is small enough that the wedge is negligible. We treat $F_0^{\text{forward}} \approx F_0^{\text{futures}}$.
- Exception: **long-term fixed-income futures** (Treasury bond, KTB, Bund). Bond prices and rates move opposite to each other, with high correlation. The wedge becomes non-trivial.
- For this course, we will use forward = futures unless explicitly stated.

Beyond cost-of-carry — advanced pricing models

- The simple formula $F_0 = S_0\, e^{(r-q+u-y)T}$ assumes deterministic rates, no jumps, perfect liquidity, and a tradable spot. Real markets violate all four.
- **Reading:** Hull Ch. 28 (martingales & measures), Ch. 35 (energy/commodities); Schwartz (1997, *J. Finance*).

Wrap-up — and a bridge to swaps

- **Today.** The cost-of-carry pricing formula

$$F_0 = S_0\, e^{(r-q+u-y)\,T},$$

  with q for dividend yield, u for storage cost, y for convenience yield.

- Forward valuation as time passes: $f = (F_t - F_0)\, e^{-r(T-t)}$ for the long; opposite sign for the short.

- Forward $\approx$ futures, except where the underlying is highly correlated with the risk-free rate (e.g. long-bond futures).

- **Lec 5 (next).** *Swaps* — a stack of forwards bundled together. Same no-arbitrage logic, applied to fixed-vs-floating cash flows.