BUSS386 Problem Set 2 — Solutions

Interest Rates

Prof. Ji-Woong Chung

Problem 1 — Compounding conversion (15 pts)

The bank quotes $r_4 = 14\%$ (quarterly compounding, $m = 4$). The future value of \$1 in one year is $(1 + 0.14/4)^4 = 1.035^4 = 1.14752$.

(a) Continuous: $e^{r_c} = 1.14752 \Rightarrow r_c = \ln(1.14752) = \boxed{13.76\%}$.

(b) Annual: $(1 + r_1) = 1.14752 \Rightarrow r_1 = \boxed{14.75\%}$.

(c) (a) is *lower* than (b). Continuous compounding is the most "frequent" compounding, so a lower stated rate produces the same effective return as a higher annual rate.

Problem 2 — Zero rates and bond pricing (20 pts)

(a) Discount each cash flow at the corresponding zero rate. Cash flows: \$2.50 at $t = 0.5, 1, 1.5$ and \$102.50 at $t = 2$.

$$\begin{aligned} P &= 2.50\,e^{-0.040(0.5)} + 2.50\,e^{-0.045(1)} + 2.50\,e^{-0.0475(1.5)} + 102.50\,e^{-0.050(2)} \\ &= 2.50(0.9802) + 2.50(0.9560) + 2.50(0.9311) + 102.50(0.9048) \\ &= 2.4505 + 2.3900 + 2.3278 + 92.7444 \\ &= \boxed{\$99.91}. \end{aligned}$$

(b) Par yield c such that price equals 100. Let $A = e^{-0.04(0.5)} + e^{-0.045(1)} + e^{-0.0475(1.5)} + e^{-0.05(2)} = 3.7721$. Setting price = 100:

$$\begin{aligned} 100 &= \tfrac{c}{2}\,A + 100\,e^{-0.05(2)} \\ 100 - 100(0.9048) &= \tfrac{c}{2}(3.7721) \\ 9.5163 &= 1.8860\,c \\ c &= \boxed{5.05\%\ \text{(cc)}}. \end{aligned}$$

(c) The 2-year zero rate is 5%; the par yield (5.05%) is essentially a duration-weighted average of the four zero rates, dominated by the 2-year rate because the principal cash flow at $t = 2$ is much larger than the coupons.

Problem 3 — Forward rates and FRAs (25 pts)

(a) $r_0(1.5, 2) = \dfrac{0.05(2) - 0.0475(1.5)}{2 - 1.5} = \dfrac{0.10 - 0.07125}{0.5} = \boxed{5.75\% \text{ (cc)}}$.

(b) $r_0(0.5, 1) = \dfrac{0.045(1) - 0.04(0.5)}{1 - 0.5} = \dfrac{0.025}{0.5} = \boxed{5.00\% \text{ (cc)}}$.

(c) For a 6-month period: $e^{0.0575(0.5)} = (1 + R_{sa}/2)$.
$e^{0.02875} - 1 = R_{sa}/2 \Rightarrow R_{sa}/2 = 0.02917 \Rightarrow \boxed{R_{sa} = 5.83\% \text{ (s.a.)}}$.

(d) FRA value (receive $R_K = 7\%$ s.a., pay floating, $L = \$10\text{M}$, $\tau = 0.5$):

$$\begin{aligned}
V &= L \times \frac{R_K - R_F}{2} \times e^{-r(2)} \\
&= 10{,}000{,}000 \times \frac{0.070 - 0.0583}{2} \times e^{-0.05(2)} \\
&= 10{,}000{,}000 \times 0.005835 \times 0.9048 \\
&= \boxed{\$52{,}808.}
\end{aligned}$$

(e) The value is positive, so receiving fixed at 7% is more attractive than the current forward (5.83% s.a.). At the current term structure you would prefer to *receive* fixed.

Problem 4 — Bond duration (20 pts)

(a) Cash flows: \$8 at $t = 1, 2, 3, 4$ and \$108 at $t = 5$. With $y = 11\%$ c.c.:

$$\begin{aligned}
P &= 8e^{-0.11} + 8e^{-0.22} + 8e^{-0.33} + 8e^{-0.44} + 108e^{-0.55} \\
&= 7.166 + 6.420 + 5.751 + 5.152 + 62.305 \\
&= \boxed{\$86.79}.
\end{aligned}$$

(b) Macaulay duration: weighted average time, weights = PV_i/P.

$$\begin{aligned}
D &= \frac{1(7.166) + 2(6.420) + 3(5.751) + 4(5.152) + 5(62.305)}{86.79} \\
&= \frac{7.166 + 12.840 + 17.253 + 20.608 + 311.525}{86.79} = \frac{369.39}{86.79} \\
&= \boxed{4.256 \text{ years}}.
\end{aligned}$$

(c) $\Delta P/P \approx -D\Delta y = -4.256 \times (-0.002) = +0.851\%$.
Estimated new price: $86.79 \times 1.00851 = \boxed{\$87.53}$.

(d) Recompute exactly at $y = 10.80\%$:

$$\begin{aligned}
P' &= 8e^{-0.108} + 8e^{-0.216} + 8e^{-0.324} + 8e^{-0.432} + 108e^{-0.540} \\
&= 7.181 + 6.446 + 5.785 + 5.194 + 62.945 = \boxed{\$87.55}.
\end{aligned}$$

Difference from duration estimate: \$87.55 vs \$87.53 — about \$0.02 (0.02%). Excellent agreement, since $\Delta y = 0.002$ is small.

Problem 5 — Duration-matched portfolios and convexity (20 pts)

(a) Values at $y = 10\%$ c.c.:

- Portfolio A: $V_1 = 2000\, e^{-0.10} = 1809.67$; $V_{10} = 6000\, e^{-1.0} = 2206.74$.
  Total $V_A = \$4{,}016.41$.
  $$D_A = \frac{1(1809.67) + 10(2206.74)}{4016.41} = \frac{23877.1}{4016.41} = \boxed{5.945 \text{ yrs}}.$$
- Portfolio B: $V_B = 5000\, e^{-0.10(5.95)} = 5000(0.5516) = \$2{,}757.80$.
  For a zero, duration = maturity, so $D_B = \boxed{5.95 \text{ yrs}}$.

Both durations are essentially 5.95 years.

(b) For $\Delta y = +0.001$: $\Delta P/P \approx -D\Delta y$, so

- Portfolio A: $-5.945 \times 0.001 = -0.5945\%$.
- Portfolio B: $-5.95 \times 0.001 = -0.5950\%$.

Essentially equal.

(c) For $\Delta y = +5\%$ (yield rises $10\% \to 15\%$):

- A: $V_1' = 2000e^{-0.15} = 1721.4$; $V_{10}' = 6000e^{-1.5} = 1338.6$; $V_A' = \$3{,}060.0$.
  $\%\Delta = (3060.0 - 4016.41)/4016.41 = \boxed{-23.81\%}$.
- B: $V_B' = 5000e^{-0.15(5.95)} = 5000e^{-0.8925} = 5000(0.4097) = \$2{,}048.5$.
  $\%\Delta = (2048.5 - 2757.80)/2757.80 = \boxed{-25.74\%}$.

Portfolio A (the barbell) loses *less* than Portfolio B (the bullet).

(d) Both portfolios have the same duration (first-order sensitivity), but Portfolio A's cash flows are spread between $t = 1$ and $t = 10$, so its price-yield curve is *more curved* (higher convexity) than Portfolio B's, whose single cash flow at $t = 5.95$ has lower curvature. For large yield moves, higher convexity reduces the loss — duration alone underestimates the price impact in both directions, but the underestimate is bigger for the more convex portfolio, so its actual loss is smaller.