Interest Rates

BUSS386. Futures and Options

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Lecture Outline

- Interest Rates
    - Types
    - Measurement
    - Zero/Par/Forward rates
    - Forward Rate Agreements
    - Duration and convexity
- Reading: Ch. 4

Interest Rates: Types and Measurement

Interest Rates — why we care

- Every derivative pays off (or is priced) at a future date $\Rightarrow$ we need a discount rate.
- Riskier investments quote higher rates. The **risk-free rate** is the benchmark.
- Today's anchors (illustrative; check current values):
  - US: Fed funds target $\approx$ 3.75–4.00%, 10Y UST $\approx$ 4.2%.
  - Korea: BoK base rate $\approx$ 2.50%, 10Y KTB $\approx$ 3.1%.

Sources: FRED `https://fred.stlouisfed.org/` BoK ECOS `https://ecos.bok.or.kr/`

Reference rates — a quick taxonomy

Type	Examples	Notes
Treasury / sovereign	UST (US), KTB (Korea)	Near risk-free; benchmark for pricing[1]
Interbank (legacy)	LIBOR (USD, GBP)	Unsecured; *phased out end of 2021*
Overnight, unsecured	Fed funds (US), SONIA, €STR	Central-bank policy anchor
Overnight, secured (repo)	SOFR (US), KOFR (Korea)	Post-LIBOR replacements
Domestic interbank	CD rate, KORIBOR (Korea)	Used in some Korean loan products

- Higher credit risk $\Rightarrow$ higher rate (Treasury $<$ secured repo $<$ unsecured interbank $<$ corporate).
- In this course, the "risk-free rate" typically means a Treasury or repo (SOFR/KOFR) rate.

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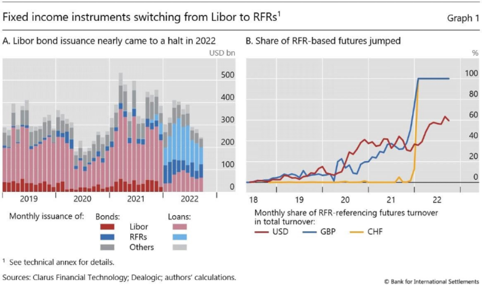
[1] Not always: see the "default trap."

LIBOR phase-out — a one-slide story

- **2012 LIBOR scandal:** banks colluded in submissions; credibility lost.
- **End of 2021:** most LIBOR tenors retired by regulators worldwide.
- Replaced by *transactions-based* overnight rates:
    - USD: **SOFR** (secured, repo)
    - GBP: SONIA EUR: **€STR**
    - JPY: TONAR CHF: SARON
    - Korea: **KOFR** (secured, repo)
- Some legacy contracts still cite LIBOR; transition is ongoing.



Fixed income instruments switching from Libor to RFRs[1]

Graph 1

A. Libor bond issuance nearly came to a halt in 2022

B. Share of RFR-based futures jumped

[1] See technical annex for details.

Sources: Clarus Financial Technology; Dealogic; authors' calculations.

Reading: BIS, "The post-LIBOR world" — `https://www.bis.org/publ/qtrpdf/r_qt2212e.htm`

Measuring Interest Rates

- Interest rates are quoted as an **annual** rate.
- However, the actual frequency at which interests are earned (or compounded) does not have to be annual
  - semiannual compounding: interests are earned in every six months.
  - quarterly compounding: interests are earned in every quarter.
  - monthly compounding: interests are earned in every month.
  - $\vdots$
- Let r denote the annual rate and m denote the number of compounding periods in a year. If you invest \$1 for one year, then future value is

$$\$1 \times \left(1 + \frac{r}{m}\right)^{m}$$

Measuring Interest Rates — continuous compounding

- **Continuous compounding**: the $m \to \infty$ limit. Interest is credited every instant.
- Future value of \$1 after T years at rate r:

$$\$1 \times e^{rT}.$$

- Why we use it in derivatives:
  - It turns multiplication-over-time into addition: $e^{rT_1} \cdot e^{rT_2} = e^{r(T_1+T_2)}$, so log-returns and rates are *additive across periods*.
  - Makes calculus clean: $\frac{d}{dT} e^{rT} = r\, e^{rT}$.
  - Black–Scholes (Lec 10) and most pricing models assume continuous rates and log-normal prices.

Conversion of Interest Rates

- Sometimes we need to convert between continuous and m-times-per-year compounding.
- Equivalence: same future value of \$1 over one year.

$$e^{r_c} \;=\; \left(1 + \frac{r_m}{m}\right)^m$$

Q. A rate is quoted as $r_2 = 10\%$ per annum with semiannual compounding. Find the equivalent continuously compounded rate r_c.

A. Solve $e^{r_c} = (1 + 0.10/2)^2 = 1.1025$:

$$r_c = \ln(1.1025) = \boxed{9.76\%}.$$

Note: $r_c < r_m$ for the same effective return — continuous compounding is “more frequent,” so a lower stated rate is enough.

NB: given prices P_t, P_{t+1}, the continuously compounded one-period rate is $r_c = \ln(P_{t+1}/P_t)$.

Zero Rates

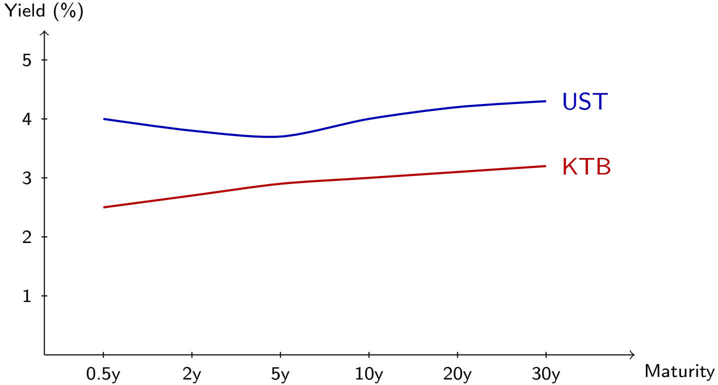
- A **zero-coupon bond** pays principal-plus-interest only at maturity (no coupons).
- Its yield is the *n*-**year zero rate** (a.k.a. *n*-year spot rate).

Q. A 5-year zero-coupon bond with principal \$1,000 is priced at \$890 today. Find the 5-year zero rate with continuous compounding.

A. Solve $890 \cdot e^{5r} = 1000$:

$$r = \frac{1}{5}\ln(1000/890) = \boxed{2.33\%}.$$

Zero rates I — shape of the yield curve



- The plot of zero rate against maturity is the **term structure** (or yield curve).
- Three canonical shapes: *normal* (rising), *flat*, *inverted* (falling).
- Levels are illustrative for early 2026 — pull a current snapshot

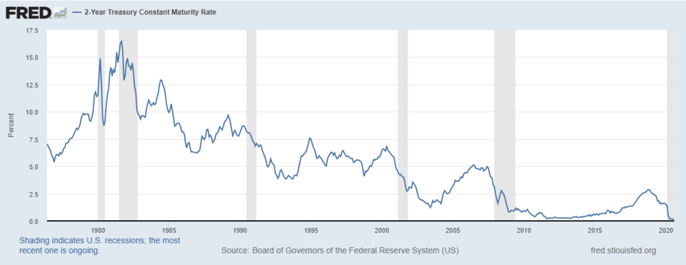
Sources: U.S. Treasury
`https://home.treasury.gov/resource-center/data-chart-center/interest-rates/`

KOFIA `https://www.kofiabond.or.kr/`

Zero rates II — the curve moves, and it forecasts

- Each zero rate is itself time-varying. The 2-year UST has ranged from ∼0.1% (2020) to >5% (2023).

2-year UST yield, 1976–present



Source: `https://fred.stlouisfed.org`

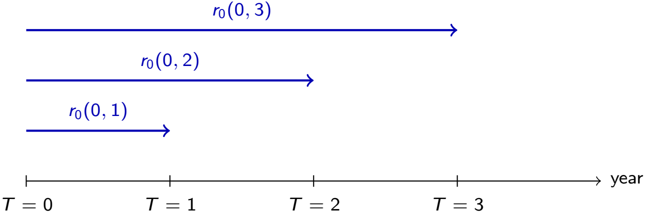
- **The shape forecasts (loosely).** The US 2y/10y spread inverted from mid-2022 through late 2024 — the longest inversion since the early 1980s — and every prior such inversion has been followed by a US recession within ∼24 months.
- Korea: BoK base rate peaked at 3.50% in early 2023; cuts began Oct 2024, currently ≈ 2.50%.

Interest rate notation

- Because rates depend on the maturity, we write

$$r_0(t_1, t_2)$$

for the rate from t_1 to t_2, observed at time 0 (continuously compounded).



Bootstrapping zero rates from observed bonds

Start with the shortest zero, then peel each longer bond one cash flow at a time.

Q. Three observed prices today (face $100, semiannual coupons):

Bond	Maturity	Coupon	Price
T-bill (zero)	0.5y	—	$97.53
T-bill (zero)	1.0y	—	$94.18
Coupon bond	1.5y	4% s.a.	$96.36

Find $r_{0.5}$, r_1, $r_{1.5}$ (continuously compounded).

A.

- $97.53 = 100\, e^{-0.5\, r_{0.5}} \;\Rightarrow\; r_{0.5} = -2\ln(0.9753) = \boxed{5.00\%}$.
- $94.18 = 100\, e^{-r_1} \;\Rightarrow\; r_1 = -\ln(0.9418) = \boxed{6.00\%}$.
- For the 1.5y bond (cash flows $2, $2, $102), discount the first two with $r_{0.5}, r_1$, then back out $r_{1.5}$:

$$96.36 = 2\, e^{-0.025} + 2\, e^{-0.06} + 102\, e^{-1.5\, r_{1.5}} \;\Rightarrow\; r_{1.5} = \boxed{6.50\%}.$$

Same procedure builds a KRW zero curve from MSB (91d, 1y) and KTB (2y, 3y).

Bond Pricing

- A 2-year bond with a principal of \$100 provides coupons at the rate of 6% per annum semiannually. The theoretical price of the bond is

$$3e^{-0.05(0.5)} + 3e^{-0.058(1.0)} + 3e^{-0.064(1.5)} + 103e^{-0.068(2.0)} = 98.39$$

Maturity (years)	Zero rate (% c.c.)
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

- A yield-to-maturity is the single discount rate that gives a bond price equal to its market price ($y = 6.76\%$)

$$3e^{-y(0.5)} + 3e^{-y(1.0)} + 3e^{-y(1.5)} + 103e^{-y(2.0)} = 98.39$$

Bond Pricing (Compounded semi-annually)

- We can also use semi-annually compounded zero rates to price the bond:

$$\frac{3}{(1+0.0506/2)}+\frac{3}{(1+0.0588/2)^2}+\frac{3}{(1+0.0651/2)^3}+\frac{103}{(1+0.0692/2)^4}=98.39$$

Maturity (years)	Zero rate (% c.c.)	Zero rate (% s.a.)
0.5	5.0	5.06
1.0	5.8	5.88
1.5	6.4	6.51
2.0	6.8	6.92

- The yield-to-maturity is $y = 6.90\%$, s.a. compounded!

$$\frac{3}{(1+y/2)}+\frac{3}{(1+y/2)^2}+\frac{3}{(1+y/2)^3}+\frac{103}{(1+y/2)^4}=98.39$$

Par Rates

- The par rate for a certain bond maturity is the coupon rate that makes the bond price equal its par value.
- Suppose that the coupon on a 2-year bond is c per annum. Using the zero rates in the previous table, the value of the bond is equal to its par value of 100 when $c = 6.87\%$.

$$\frac{c}{2}e^{-0.05(0.5)} + \frac{c}{2}e^{-0.058(1.0)} + \frac{c}{2}e^{-0.064(1.5)} + \left(100 + \frac{c}{2}\right)e^{-0.068(2.0)} = 100$$

- Useful for benchmarking and pricing.
  - Can use to set the coupon rate of new bonds. (In fact, most bonds are par/coupon bonds)
  - Can compare bonds with different coupon rates.

Forward Rates

e.g. Term-structure of zero rates today:

Maturity (years)	Zero rate (% c.c.)
1	3.0
2	4.0

- The **implied forward rate** from year 1 to year 2: solve

$$e^{0.04\times 2} = e^{0.03} \cdot e^{r} \;\Rightarrow\; \boxed{r = 5.00\%}.$$

Common misconception

The forward rate is **not** the market's prediction of next year's 1-year rate.
It is the rate at which arbitrage between (i) lending 2 years and (ii) lending 1 year then rolling forward is impossible. Forecasts and forward rates are different objects.

Forward Rates - Borrowing/Lending

- We can lock in interests for future borrowing/lending at this forward rate.
- Suppose that we want to borrow \$1 one year from now and repay in year 2. Also, we want to fix now the interest rate for this borrowing.
- If we fix at the forward rate, this borrowing will have cash flows as follows:

Action	year 0	year 1	year 2
borrowing at forward rate	0	1	$-e^{0.05\times 1}$

Forward Rates - Borrowing/Lending

- We may not have a financial instrument where we can borrow or lend at the forward rate in the markets.

- Then, we can construct this by combining two zero coupon bonds. In this construction, we can choose the face-value of zero coupon bonds as we like.

Action	year 0	year 1	year 2
buy 1-yr bond (lend)	$-e^{-0.03\times1}$	1	0
sell 2-yr bond (borrow)	$e^{-0.03\times1}$	0	$-e^{-0.03\times1}e^{0.04\times2}$
net	0	1	$-e^{0.05\times1}$

Forward Rates - Example

- Consider the following term-structure of interest rates:

Maturity (years)	Interest rate (%)
1	3.0
2	4.0
3	4.6

Q1. What is the implied forward rate $r_0(2,3)$?

Forward Rates - Borrowing/Lending - Example

Q2. An investor wants to lend \$100 in year 2 and receive in year 3. The investor wants to lock in the interest rate at $r_0(2, 3)$. Construct this instrument using two zero bonds.

Forward Rate Agreements (FRA)

- An **FRA** is an agreement to exchange a fixed rate for a floating reference rate observed at a future date. The principal itself is *not* exchanged — only the net interest.
- Today's FRAs typically reference **3-month SOFR** (USD) or **3-month KOFR** (KRW), not LIBOR.

Example. Two parties agree to exchange a fixed 3% for 3-month SOFR in 2 years, on \$100M notional, quarterly compounding. If 3-month SOFR turns out to be 3.5% at fixing:

$$\text{Net to fixed-receiver} = \$100{,}000{,}000 \times (0.035 - 0.030) \times 0.25 = \boxed{\$125{,}000.}$$

Reading: BIS, "The Post-LIBOR World" — `https://www.bis.org/publ/qtrpdf/r_qt2212e.htm`

FRA — valuation as time passes

- At inception, the FRA fixed rate is set so the contract has zero value ⇒ FRA fixed rate = forward rate.
- As markets move, the forward rate changes and the FRA value deviates from zero.

Example. Today's forward SOFR for 1.5y–2y is 5.0% (s.a. comp). A company entered an FRA earlier locking 5.8% (receive fixed, pay SOFR) on \$100M for that period. The 2-year SOFR risk-free rate is 4% (continuous comp).

A. Value to the fixed-receiver:

$$V = \$100{,}000{,}000 \times \frac{0.058 - 0.050}{2} \times e^{-0.04(2)} = \boxed{\$369{,}200.}$$

Intuition: locked rate (5.8%) is now *above* the market forward (5.0%) ⇒ receive-fixed leg is in the money.

Settlement convention. For LIBOR-based FRAs (legacy), the reference rate was set at the start of the loan period, so contracts cash-settled at T_1 with the gain discounted by the realised fixing rate to T_2. SOFR is *backward-looking* (compounded daily in arrears), so the reference isn't known until T_2; modern SOFR FRAs cash-settle at T_2 with no discounting — the rate has already accrued.

Interest Rate Risk: Measurement and Management

- Yield volatility $\Rightarrow$ price volatility for fixed-income securities.
- Banks, asset managers, and pension funds carry large interest-rate exposure.
- They manage it with futures, swaps, and dynamic hedging — much of the rest of this course.

Measurement: Duration and Convexity

- Bond price is a function of yield, $P(y)$, plus other factors (maturity, coupon, embedded options, credit, market conditions).
- For cash flows K_i at times t_i:

$$P_0 = \sum_{i=1}^{N} K_i \, e^{-y_c t_i} \text{ (c.c.)}, \quad P_0 = \sum_{i=1}^{N} K_i \, (1 + y_m/m)^{-m t_i} \; (m/\text{yr}).$$

- Two derivatives summarize how P moves with y:
  - $\partial P/\partial y \rightarrow$ **duration** (slope of the price–yield curve).
  - $\partial^2 P/\partial y^2 \rightarrow$ **convexity** (curvature).

Why we bother

Repricing a bond from scratch every time the yield moves is slow. Duration + convexity gives a fast, accurate Taylor approximation: *slope* + *curvature* of the same curve. That's the entire fixed-income risk toolkit in two numbers.

Duration: Basics

- Duration measures **first-order** bond-price sensitivity to interest-rate changes. Higher duration ⇒ higher price volatility.
- Macaulay duration[2]:
  $$D_{Mac} = -\frac{dP/P}{d(1+y/m)/(1+y/m)} = -\frac{dP/P}{dy}\,(1+y/m).$$
- Computed from *promised* cash flows — accurate only for risk-free bonds with no embedded options.
- Duration changes over time (and as y moves).

Working intuition — "zero-equivalent maturity"

A coupon bond with duration D has the *same* first-order rate sensitivity as a D-year zero-coupon bond at the same yield.
So a 4-year-duration portfolio behaves like a 4-year zero for small yield moves — which is why portfolio managers quote risk in years of duration.

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[2] Introduced by F. R. Macaulay (1938).

Duration — the formula and the picture in your head

- Macaulay duration:

$$D_{Mac} = \sum_{i=1}^{N} t_i \cdot \underbrace{\frac{K_i\, e^{-y_c t_i}}{P}}_{\text{weight}_i}; \qquad D_{Mac} = \sum_{i=1}^{N} t_i \cdot \underbrace{\frac{K_i\,(1+y_m/m)^{-m t_i}}{P}}_{\text{weight}_i}$$

Treat each cash flow's PV as a *weight*. Stack the weights along a time-line at $t_1, t_2, \ldots, t_N$. Duration is the **center of mass** (the balance point) of those weights.
For a zero ($N = 1$): center of mass = maturity. For a coupon bond, coupons earlier than maturity pull the balance point in $\Rightarrow D <$ maturity.

Four facts to remember:

- Coupon bond: $D \leq$ maturity.
- Zero: $D =$ maturity.
- Higher coupon $\Rightarrow$ shorter D (more weight on earlier coupons).
- Higher yield $\Rightarrow$ shorter D (distant cash flows discounted away).

Modified Duration

- Modified duration:

$$D_{Mod} = \frac{D_{Mac}}{1 + y/m} \quad \Longrightarrow \quad D_{Mod} = -\frac{dP/P}{dy}.$$

- Special case (continuous compounding): $D_{Mod} = D_{Mac}$ (the $1/(1 + y/m)$ adjustment vanishes as $m \to \infty$).

- First-order approximation:

$$\boxed{\frac{\Delta P}{P} \approx -D_{Mod}\,\Delta y}$$

If $D_{Mod} = 5$ years and the yield rises by 1% (100 bp), the bond loses about $-5 \times 0.01 = -5\%$ of its price.
That's the working number a portfolio manager actually quotes. Macaulay duration is the same idea "in time units"; modified duration translates it into "percent price per percent yield."

Caveat: this is only the slope. For larger Δy (say more than 50–100 bp), the curvature of $P(y)$ matters — that's convexity, coming up.

Dollar Duration

- **Dollar duration** $D_d = D_{Mod} \times P = -\frac{dP}{dy}$.
- So $\Delta P \approx -D_d \, \Delta y = -D_{Mod} \, P \, \Delta y$.

Why traders prefer this

Modified duration tells you a *percentage*; dollar duration tells you a *dollar amount*. If a bond has $D_d = \$5{,}000$ per 1% of yield, a 1 bp move is worth \$50. That's the unit a trading desk hedges in — you can't add 5% of bond A to 5% of bond B, but you *can* add their dollar durations.

- Especially useful for *zero-NPV portfolios*: there is no P to take a percentage of, but the dollar sensitivity to yield is well-defined.

Portfolio Duration

- Modified duration of a portfolio = value-weighted average of the components':

$$D_{Mod}^{\text{port}} = \sum_i w_i\, D_{Mod}^{(i)}, \quad w_i = \frac{P_i}{\sum_j P_j}.$$

- Dollar duration of a portfolio = simple sum:

$$D_d^{\text{port}} = \sum_i D_d^{(i)}.$$

- Zero-value portfolio (e.g. swap): no P to weight by, so only dollar duration is defined.

Example

\$10M of duration-5 bonds + \$5M of duration-10 bonds.
$D_{Mod}^{\text{port}} = \frac{10}{15}(5) + \frac{5}{15}(10) = 6.67$ years.
$D_d^{\text{port}} = \$10\text{M}(5) + \$5\text{M}(10) = \$100\text{M}$ of dollar duration
(i.e., a 1% rate rise costs ≈ \$1M).

Effective Duration — when cash flows are not fixed

- For *option-free* bonds, modified duration is exact. But callable bonds, MBS, and credit-risky bonds have *uncertain cash flows* — the formulas based on promised K_i are inaccurate.
- **Effective duration** sidesteps the formula: shock the yield up and down by x basis points, observe the actual price change.

$$D_{eff} = -\frac{1}{P}\,\frac{P(y_0 + x) - P(y_0 - x)}{2x}.$$

Why this matters

Take a callable bond. When rates fall, the issuer is more likely to call the bond and refinance — so the bond's price stops rising as fast as a regular bond would. Modified duration, computed on the promised cash flows, would miss that. Effective duration captures it because it uses an actual pricing model that knows about the call option.

Mortgage-backed securities use effective duration almost exclusively because prepayments make their cash flows highly state-dependent.

Effective duration — example (callable bond)

5-year callable corporate bond, \$100 face, 5% s.a. coupon, callable at \$100 from year 2 on. $y_0 = 5\%$, $P_0 =$ \$100. Reprice for ±25 bp shocks; callable price $=$ straight price $-$ value of issuer's call.[3]

Scenario	Yield	Straight	Call value	Callable
Down	4.75%	\$101.10	\$0.60	\$100.50
Base	5.00%	\$100.00	\$0.00	\$100.00
Up	5.25%	\$98.91	\$0.00	\$98.91

$$D_{eff} = -\frac{1}{100} \cdot \frac{98.91 - 100.50}{2(0.0025)} = \boxed{3.38 \text{ yrs}} \quad \text{vs.} \quad D_{Mod} \approx 4.42 \text{ yrs (naive).}$$

The call gets more valuable as rates fall, capping price on the upside. So $D_{eff} < D_{Mod}$ for callable bonds, MBS, and credit-risky bonds — D_{eff} is the right number for hedging.

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[3] Straight prices computed from $\sum 2.50/(1+y/2)^t + 100/(1+y/2)^{10}$. Call values illustrative; exact valuation uses a binomial-tree model (Lec 9).

Convexity — the curvature of the price–yield curve

- **Convexity** = curvature of the price–yield relation (the second-order term).
- Long positions in non-callable bonds have *positive* convexity.
- Positive convexity is desirable: duration alone *understates* the price gain when yields fall and *overstates* the price loss when yields rise.

Convexity — the formula

- Convexity is the second derivative of price with respect to yield, normalized by price:

$$C_{Mod} = \frac{1}{P}\,\frac{d^2P}{dy^2} \;=\; \sum_{i=1}^{N} t_i^2\,\frac{K_i\,e^{-y_c t_i}}{P} \quad \text{(c.c.)}.$$

$$C_{Mod} = \sum_i t_i\left(t_i + \tfrac{1}{m}\right)\frac{K_i(1+y_m/m)^{-(mt_i+2)}}{P} \quad \text{(m/yr)}.$$

- Dollar convexity: $C \times P$.

The picture (compare to duration)

Duration is a weighted average of t_i. Convexity is a weighted average of t_i^2. Squaring puts *much* more weight on distant cash flows. Two bonds with identical duration but different maturity profiles can have very different convexity — the one with cash flows spread further apart wins.

Units are years^2 — they don't mean anything intuitive on their own.

Putting it together — duration + convexity

Just a Taylor expansion of $P(y)$

$$P(y) - P(y_0) \;\approx\; P'(y_0)\,(y - y_0) \;+\; \tfrac{1}{2}\,P''(y_0)\,(y - y_0)^2.$$

With $P'(y_0) = -D \cdot P$, $P''(y_0) = C \cdot P$, this becomes:

$$\boxed{\frac{\Delta P}{P} \;\approx\; -\,D\,\Delta y \;+\; \tfrac{1}{2}\,C\,(\Delta y)^2.}$$

- Duration term: *slope* of the price–yield curve at y_0.
- Convexity term: *curvature* correction. Always raises the estimate of ΔP (positive convexity) — so it cushions losses on rate rises and adds to gains on rate falls.

Rough rule of thumb

For $|\Delta y| < 50$ bp, duration alone is fine.
For $50 \leq |\Delta y| < 200$ bp, convexity matters.
For $|\Delta y| > 200$ bp, even the second-order Taylor expansion gets shaky — reprice the bond.

Hedging with Duration and Convexity

- Change in bond price with a change in interest rates:

$$\Delta P \approx -DP\Delta y + \frac{1}{2}CP\Delta y^2$$

- A delta neutral portfolio equates the hedge ratio (duration) of assets and liabilities.

- A gamma neutral portfolio is delta neutral, and also equates the gammas (convexity) of assets and liabilities.

Example — a 2026 dealer hedging story

Setup. End of trading day. A corporate-bond dealer is stuck with \$10M of a 5-year corporate bond, priced at par, currently yielding $y_C = 5.0\%$ s.a. The bond is illiquid; she wants to neutralize overnight rate risk by shorting U.S. Treasuries.

Available UST hedge instruments:

- 10-year UST at $y_{10} = 4.20\%$ s.a.; modified duration $D_{10} = 8.05$ yrs.
- 3-year UST at $y_3 = 3.80\%$ s.a.; modified duration $D_3 = 2.85$ yrs.
- Inventory bond modified duration: $D_C = 4.40$ yrs.

Questions.

(a) How much 10-year UST should she short to be duration-neutral? How much 3-year UST?

(b) If *all* yields rise by 100 bp overnight, what is the P&L of (i) the unhedged position, (ii) the hedged position?

Example (cont'd) — duration matching

(a) Dollar-duration match. Set $D_C \cdot V_C = D_{\text{UST}} \cdot X$:

- Hedge with 10y UST alone: $X_{10} = \$10\text{M} \times \frac{4.40}{8.05} = \boxed{\$5{,}466{,}000}$ to short.
- Hedge with 3y UST alone: $X_3 = \$10\text{M} \times \frac{4.40}{2.85} = \boxed{\$15{,}439{,}000}$ to short.

(b) P&L if all yields rise 100 bp ($\Delta y = +0.01$). Using $\Delta P/P \approx -D\Delta y$:

- Inventory bond loss: $4.40 \times 0.01 \times \$10\text{M} = \$440{,}000$.
- 10y UST short gain: $8.05 \times 0.01 \times \$5{,}466{,}000 \approx \$440{,}000$.
- 3y UST short gain: $2.85 \times 0.01 \times \$15{,}439{,}000 \approx \$440{,}000$.

Either hedge offsets the first-order loss. The two hedges differ in *convexity* (and curve risk): the 10y UST has higher convexity but loads on the long end; the 3y UST loads on the short end and is barely convex. We see the difference next slide.

Example (cont'd) — gamma-neutral hedge

Idea. For protection against *larger* yield moves, also match convexity. Using both UST instruments:

- Inventory: $V_C = \$10\text{M}$, $D_C = 4.40$, $C_C = 23$.
- 10y UST: $D_{10} = 8.05$, $C_{10} = 78$.
- 3y UST: $D_3 = 2.85$, $C_3 = 9.5$.

Two equations in two unknowns (X_{10}, X_3):

$$\begin{aligned} \text{Duration:} \quad & 8.05\, X_{10} + 2.85\, X_3 = 4.40 \times \$10\text{M} \\ \text{Convexity:} \quad & 78\, X_{10} + 9.5\, X_3 = 23 \times \$10\text{M} \end{aligned}$$

Solution: $\boxed{X_{10} \approx \$1.63\text{M}, \quad X_3 \approx \$10.84\text{M}}$ (both shorted). The barbell of long-end and short-end USTs replicates both the slope and the curvature of the dealer's price-yield curve.

Wrap-up — and a bridge to swaps

- **Today.** Reference rates (Treasury, SOFR, KOFR), compounding conventions, the zero curve, par and forward rates, FRAs, duration, and convexity.
- **Big idea.** Every fixed-income (and many derivative) cash flows can be valued by discounting along the zero curve; sensitivity to that curve is summarized by duration and convexity.
- **Why this matters next week.** The whole forwards/futures pricing machinery (Lec 3–4) rests on the no-arbitrage cost-of-carry, which uses exactly the rates from today.
- **Where rate hedging actually happens.** In practice, the dealer hedge we just built with cash bonds is overwhelmingly done with **interest-rate swaps**. We construct, price, and hedge swaps in **Lec 5**.

Reading for next class: Hull, Ch. 5.