Practice Problem: Solution

BUSS386 Futures and Options

1 Barrier Options

What is the price of a down-and-out put when the barrier is greater than the strike price?

Answer: The option is in the money only when the asset price is less than the strike price. However, in these circumstances the barrier has been hit and the option has ceased to exist

2 Forward-Start Options

Suppose that $S = \$100$, $\sigma = 30\%$, and $r = 8\%$. Today you buy a contract which, 6 months from today, will give you one 3-month to expiration at-the-money call option. (This is called a forward start option.) Assume that r and σ are certain not to change in the next 6 months.

a Six months from today, what will be the value of the option if the stock price is \$100? \$50? \$200? (Use the Black-Scholes formula to compute the answer.) In each case, what fraction of the stock price does the option cost?

b What investment today would guarantee that you had the money in 6 months to buy an at-the-money option?

c What would you pay today for the forward start option in this example?

Answer:
a. In 6 months,
$c_6(100) = 6.962$, fraction= 0.0696
$c_6(50) = 3.481$, fraction= 0.0696
$c_6(200) = 13.924$, fraction= 0.0696

b. Regardless of the stock price in 6 months, the value of the ATM call option is $0.0696 * S_6$. Buy 0.0696 unit of the stock: $0.0696 \times S0 = 0.0696 \times \$100 = \$6.96$
.

c. You would therefore pay \$6.96 for this option.

3 Monte Carlo Pricing of Asian Options

Estimate the value of a 6-month European-style (arithmetic) average price call option on a non-dividend-paying stock. The initial stock price is \$30, the strike price is \$30, the risk-free rate is 5%, and the stock price volatility is 30%.

a Run a Monte Carlo simulation to estimate the price of the option. Use 1,000 stochastic paths for the risk-neutral representation of the evolution of stock prices assuming lognormality, drawing innovations from a normal distribution, and with a time step h equal to 1 month.

b Repeat this exercise but now set the time step h equal to 1 week (treating 6 months as 26 weeks).

c Using the same Monte Carlo simulation of stock prices as in Part (b), what is the price of a knock-in call option with a strike price of \$30 and a barrier of \$35?

Hint: We can simulate the lognormal stock price process under the risk-neutral representation using the following algorithm:

$$S_{t+h} = S_t e^{(r-\sigma^2/2)h+\sigma\sqrt{h}\epsilon_t},$$

where the time step $h = \frac{1}{12}$ and $\epsilon_t \sim N(0,1)$.

After generating 1,000 paths for the stock price, we can calculate the payoff on each path i as:

$$V_i = \max\left(\frac{1}{6}\sum_{t=1}^{6} S_t^i - K, 0\right)$$

Finally, the value of the average price call is given by $c = e^{-0.05(0.5)}\frac{1}{1000}\sum_i V_i$.

4 Monte Carlo Valuation of Knock-In Option

The knock-in call option comes into existence when the stock price reaches \$35 before expiration. Using the same algorithm to simulate the lognormal stock price process as in Part (b) with $h = 1/52$, we can simulate the payoff of the knock-in call on a particular path as $\max(S_T - K, 0)$ if the stock price reaches \$35, and 0 otherwise.

Closed-form formula (Optional): The option's payoff depends on the arithmetic average of the price of the underlying stock during the life of the option. In particular, the payoff is $\max(0, A(T) - K)$, where $A(T)$ is the arithmetic average price of the stock. Under the assumption that $A(T)$ is lognormally distributed, the average price call can be valued using a similar formula to the one we have used to price a regular European call. Suppose M_1 and M_2 are the first two moments of $A(T)$. The value of the average price call is given by Black's model:

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$$
$$d_1 = \frac{\ln(F_0/K) + (\sigma^2/2)T}{\sigma_F\sqrt{T}}$$
$$d_2 = d_1 - \sigma_F sqrtT,$$

where $F_0 = M_1$ and $\sigma_F^2 = \frac{1}{T}\ln\frac{M_2}{M_1^2}$ Assuming that the average is calculated continuously,

$$M_1 = \frac{e^{(r-q)T} - 1}{(r-q)T} S_0$$
$$M_2 = \frac{2e^{[2(r-q)+\sigma^2]T S_0^2}}{(r-q+\sigma^2)(2r-2q+\sigma^2)T^2} + \frac{2S_0^2}{(r-q)T^2}\left(\frac{1}{2(r-q)+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right).$$

Plugging in $r = 5\%, q = 0, \sigma = 30\%, T = 0.5, S_0 = 30, K = 30$ to the expressions for M_1 and M_2 above, we get that $M_1 = 30.378, M_2 = 936.9, \sigma_F^2 = 17.41$. Therefore,

$$c = e^{-0.05(0.5)}[30.378N(0.163) - 30N(0.04)] = 1.64$$