Option Greeks

BUSS386. Futures and Options

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Lecture Outline

- The Greeks
- Applications
  - Capital protection products
  - Risk management
- Disclamer: the discussion is based on the BSM model. The empirical (actual) values of the Greeks are usually different from what the BSM model predicts!

"The Greeks"

- The sensitivity of option value to various factors
- They are known as "The Greeks."
  1. Delta
  2. Gamma
  3. Theta
  4. Vega
  5. Rho
- They are used for risk management as well as trading.
- For options that can be priced using BSM, they often take a simple form.

Delta, Δ

1. Delta: Sensitivity of option to changes in the underlying price.

$$\Delta = \frac{\partial V}{\partial S} = N(d_1) \text{ for Calls}$$

   - For dividend paying underlyings: $e^{-q(T-t)} N(d_1)$
   - For put: $N(d_1) - 1$. With dividends: $e^{-q(T-t)}(N(d_1) - 1)$

- It tells how many units of the underlying asset one should trade in order to hedge the market risk exposure of the option.
  - For example, if $\Delta = 0.50$ for a given call option, the position that is long one call and short 0.50 shares of stock will be hedged against a (small) change in the stock price up or down (Delta neutral hedge)
  - Delta measures **market risk**.

- Approximately the probability that an option finishes in the money (in a risk neutral world).
  - $\text{Prob}\,(S_T \geq K) = N(d_2) = N(d_1 - \sigma\sqrt{T}) \propto N(d_1)$

Delta, Δ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

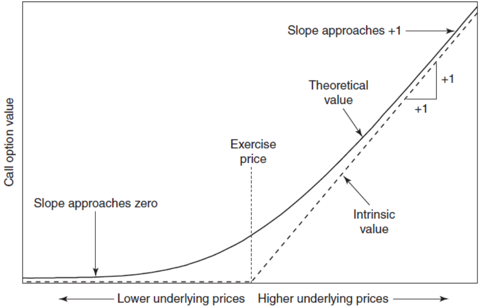
$$d_1 = \frac{ln(49/50) + (0.05 + 0.2^2/2)0.3846}{0.2 \times \sqrt{0.3846}} = 0.0542$$

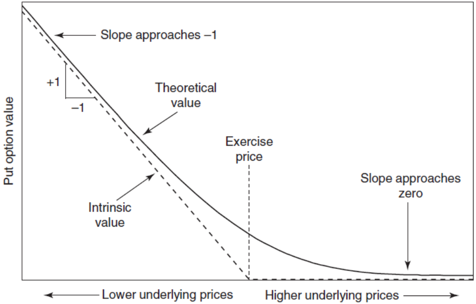
- Delta is $N(d_1) = 0.522$. When the stock price changes by ΔS, the option price changes by $0.522\Delta S$.

Delta, Δ (cont'd)

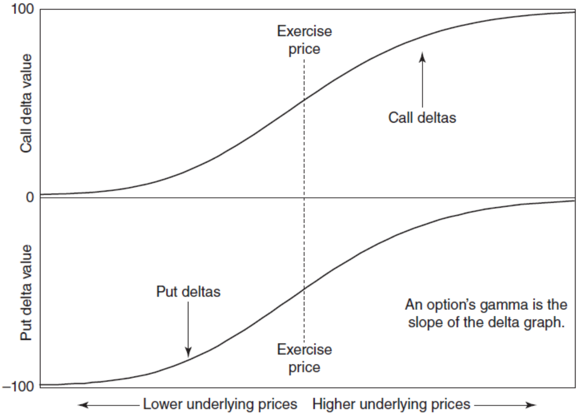
- The delta of is positive for calls and negative for puts.
- The delta is close to ± 1 for deep in the money options.
- The delta of far out of the money option is close to 0.
- At the money option has delta of about ± 0.50.

Delta, Δ (cont'd)





Delta, Δ (cont'd)



Same shape for call and put: the put-call parity!

Gamma, Γ

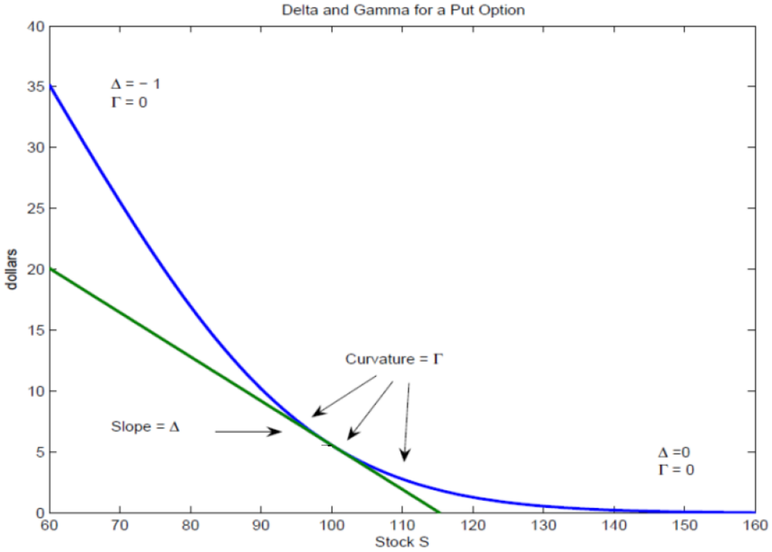
2. Gamma: Sensitivity of Delta to changes in the underlying price.

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$

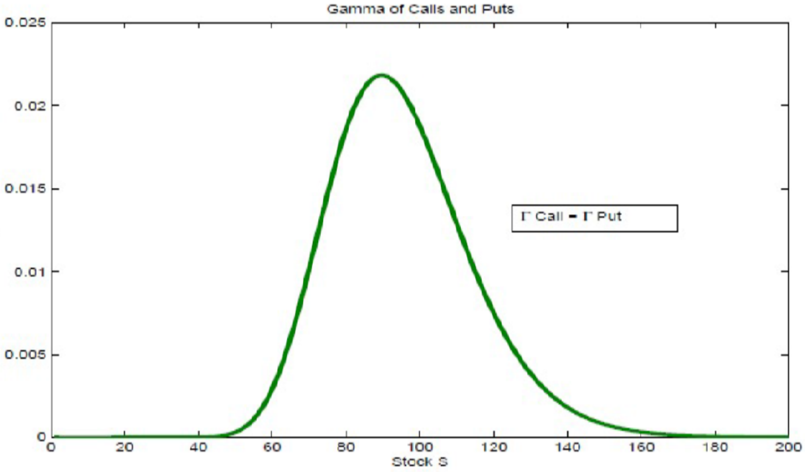
where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

- Identical for both calls and puts. For dividend paying underlyings?
- Gamma measures risk for a delta neutral hedge.
- Gamma is related to the curvature of the option value function.
  - For a long position, always positive.
  - Gamma is the largest at the money.
  - Gamma is small in the deep in or out of the money.

Gamma, Γ (cont'd)

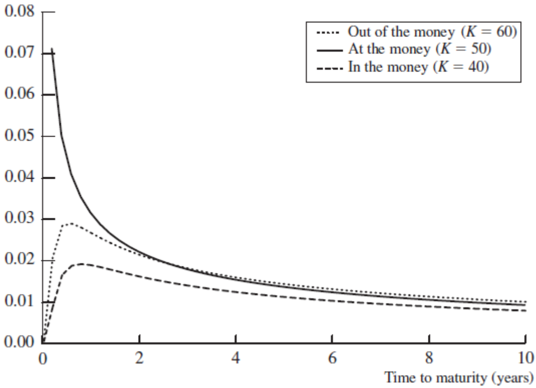


Gamma, Γ (cont'd)



Gamma, Γ (cont'd)

Variation of gamma with time to maturity for a stock option ($S = 50, r = 0, \sigma = 25\%$).



For an at-the-money option, gamma increases as the time to maturity decreases. Short-life at-the-money options have very high gammas, which means that the value of the option holder's position is highly sensitive to jumps in the stock price.

Gamma, Γ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$\frac{N'(d_1)}{S\sigma\sqrt{T}} = 0.066$$

- When the stock price changes by ΔS, the delta of the option changes by $0.066\Delta S$.

Gamma, Γ (cont'd)

- A call has a Delta of 0.54 and Gamma of 0.04.
  - Stock goes up \$1: Delta will become more positive by the Gamma amount.
  - New Delta value: 0.58

- Another call has a Delta of 0.75 and Gamma of 0.03
  - Stock is down \$1: Delta will become less positive by Gamma amount.
  - New Delta value: 0.72

- XYZ: $S = \$50, K = \$50, C = \$2, \Delta = 0.50, \Gamma = 0.06$
  - Should XYZ go up to \$51, the 50 strike call will be worth around \$2.50 when using delta only.
  - Using gamma as well: $c(51) - c(50) = \Delta(51 - 50) + \frac{1}{2}\Gamma(51 - 50)^2$
  - Delta $\approx$ (Dollar) duration, Gamma $\approx$ (Dollar) convexity

Theta (Θ)

- **Theta measures "time decay."** As time passes, the option's extrinsic value melts away.
- **Call and put thetas are usually negative.** Options lose value as time passes (holding all else constant).

Theta, Θ

3. Theta: Sensitivity of option to passage of time, t.

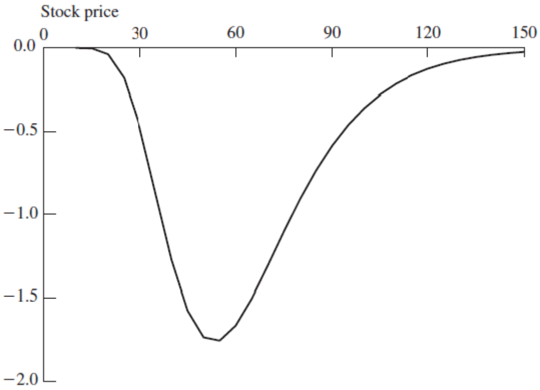
$$\Theta = \frac{\partial V}{\partial t} = \begin{cases} -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) & \text{for Calls} \\ -\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) & \text{for Puts} \end{cases}$$

where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

- Theta measures **Time Decay**.
- Theta decreases (more negative) when the option closer to expiration and at the money.
- Deep ITM or deep OTM options have low theta because little time value remains.

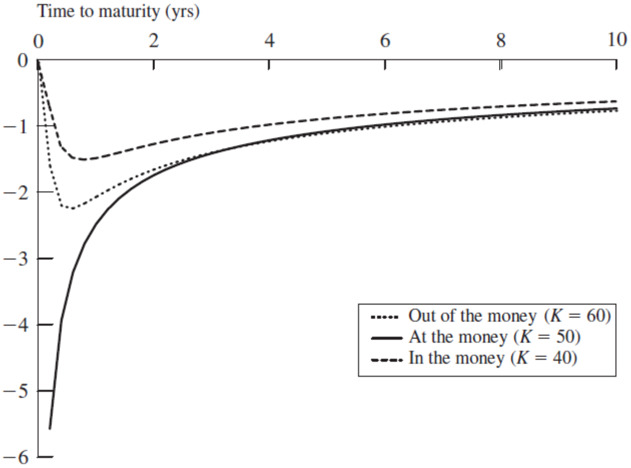
Theta, Θ (cont'd)

European call option $K = 50, r = 0, \sigma = 25\%, T = 2$



Theta, Θ (cont'd)

European call option $S_0 = 50$, $K = 50$, $r = 0$, $\sigma = 25\%$



Theta (Θ) (Cont'd)

(Assume long option positions throughout.)

- **Call on a non-dividend-paying stock:** $\Theta < 0$
  - As time passes (with S fixed), the **variance of S_T shrinks**. A narrower distribution reduces the value of optionality.
  - The strike price K is like a **debt due at maturity**. As time passes, the discount factor $e^{-r(T-t)}$ becomes smaller $\rightarrow$ the present value of the "debt" rises. This hurts a long call holder.

- **Call on a dividend-paying stock:** Θ **can be positive**
  - A call holder does not receive dividends. If a dividend is paid soon, the stock price drops on ex-div date $\rightarrow$ bad for the call.
  - As time passes and the dividend is avoided (or gets closer to being avoided), the call's relative value can **increase**, making $\Theta > 0$ possible.
  - Deep OTM calls have very low sensitivity to this effect $\rightarrow$ Theta remains near zero.

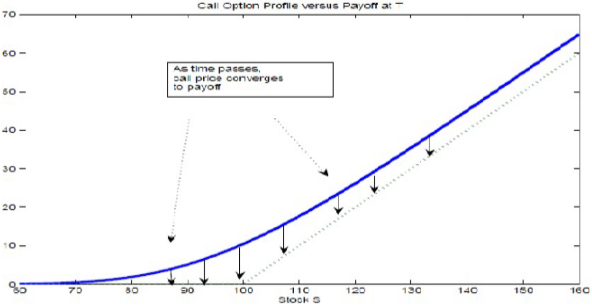
Theta (Θ) (Cont'd)

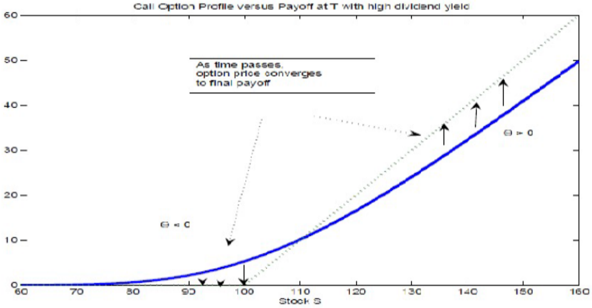
(Assume long option positions throughout.)

- **Put options: Θ can be positive or negative**
  - When S is high (put is OTM): payoff at maturity is likely zero but the put currently has time value. As time passes (with S fixed), this time value decays $\rightarrow \Theta < 0$.
  - When S is very low (deep ITM): payoff is approximately K with probability near 1. The present value of K is $Ke^{-r(T-t)}$, which **increases** as t increases (less discounting). Hence $\Theta > 0$.

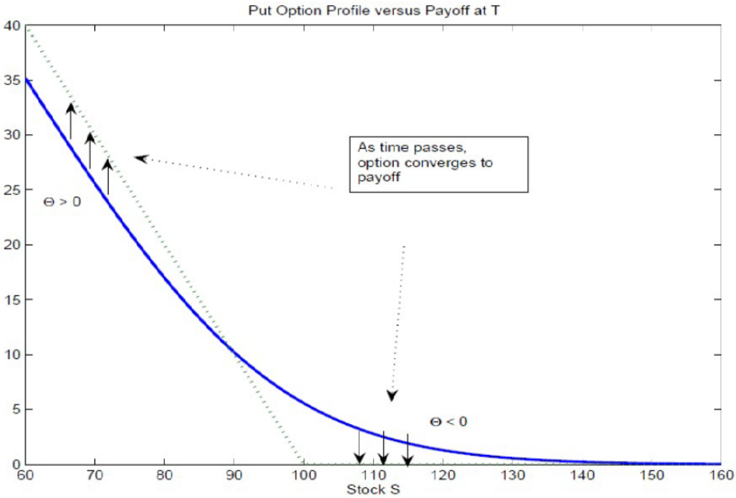
- **American options: Θ is typically negative**
  - Early exercise rights add value, but this value also declines as time passes.
  - The option still loses time value overall $\rightarrow$ **Theta is almost always negative**.

Theta, Θ (cont'd)





Theta, Θ (cont'd)



Theta, Θ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$-\frac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) = -4.31$$

- The theta is $-4.31/365 = -0.0118$ per calendar day, or $-4.31/252 = -0.0171$ per trading day.

Vega, ν

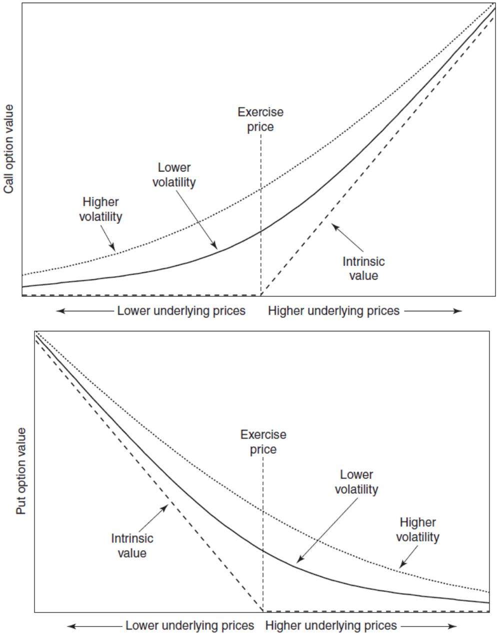
4. Vega: Sensitivity of option to a change in volatility σ.

$$\nu = \frac{\partial V}{\partial \sigma} = S\sqrt{T}N'(d_1) > 0$$

where $N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, a PDF for a standard normal distribution.

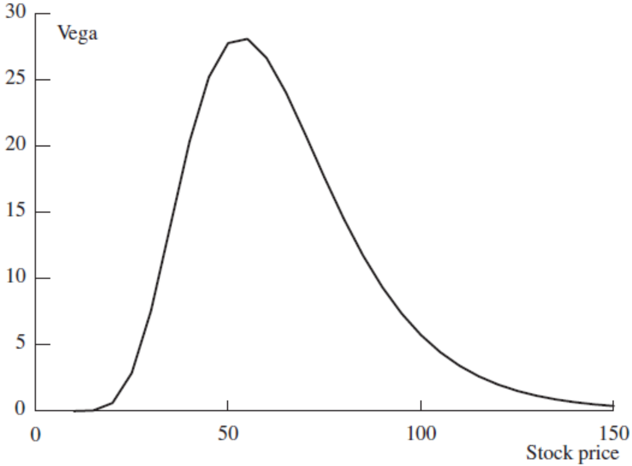
- Vega measures exposure to **Volatility Risk**
- The vega of European and American calls and puts is positive.
- For very deep OTM or ITM options, the vega is close to zero.
- The vega of a call or put peaks near the money.
- Buying a portfolio with positive vega is “buying volatility”. Typically we do this by buying a call and a put — a straddle.

Vega, ν (cont'd)



Vega, ν (cont'd)

Variation of vega with stock price for an option $K = 50$, $r = 0$, $\sigma = 25\%$, $T = 2$



Vega, ν (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$S\sqrt{T}N'(d_1) = 12.1$$

- Thus a 1% (0.01) increase in the implied volatility from (20% to 21%) increases the value of the option by approximately $0.01 \times 12.1 = 0.121$.

Rho, ρ

5. Rho: Sensitivity of option to a change in the interest rate.

$$\rho = \frac{\partial V}{\partial r} = \begin{cases} KTe^{-rT}N(d_2) > 0 & \text{for Calls} \\ -KTe^{-rT}N(-d_2) < 0 & \text{for Puts} \end{cases}$$

- Rho measures exposure to Interest Rate Risk.
- It depends on whether the option holder will pay K (call) or receive K (put). The PV of K declines as r increases, making the payment made smaller for the long call and payment received smaller for the long put.

Rho, ρ (cont'd)

- Consider a call option on a non-dividend paying stock, where $S_0 = 49, K = 50, r = 0.05, \sigma = 0.20$, $T = 0.3846$.

$$KTe^{-rT}N(d_2) = 8.91$$

- This means that a 1% (0.01) increase in the risk-free rate (from 5% to 6%) increases the value of the option by approximately $0.01 \times 8.91 = 0.0891$.

Summary

Table 19.6 Greek letters for European options on an asset providing a yield at rate q.

Greek letter	*Call option*	*Put option*
Delta	$e^{-qT} N(d_1)$	$e^{-qT}[N(d_1) - 1]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})$ $+ qS_0N(d_1)e^{-qT} - rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/(2\sqrt{T})$ $- qS_0N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Exercise

- Here are the current market prices for XYZ stock and two XYZ options. The Greek letter risk exposures come from the Black-Scholes model. The interest rate is 8% and the implied volatility is 0.25.

	Market price	delta	gamma	vega	theta
XYZ Stock	100	1	0	0	0
XYZ Call 105 strike, 1 month	1.25	0.29	0.047	.099	-.044
XYZ Put 95 strike, 1 month	0.83	-0.21	0.039	.084	-.030

- You are long 105-strike calls on 100,000 shares. (That is, you have 100,000 call options, each covering one share.)
  1. How would you set up a delta hedge for this position?
  2. What would the overall hedged position be worth? (What is the net cost to set it up?)
  3. What are the Greek letter exposures for the overall position?

Exercise (cont'd)

	Market price	delta	gamma	vega	theta
XYZ Stock	100	1	0	0	0
XYZ Call 105 strike, 1 month	1.25	0.29	0.047	.099	-.044
XYZ Put 95 strike, 1 month	0.83	-0.21	0.039	.084	-.030

1. Position delta is $100,000 \times 0.29 = 29,000$. Hedge by shorting 29,000 shares.
2.

$$\begin{aligned} Calls &= 100,000 \times 1.25 = 125,000 \\ Stocks &= -29,000 \times 100 = -2,900,000 \\ Total &= -2,775,000 \end{aligned}$$

3.

$$\begin{aligned} Delta &= 100,000 \times 0.29 + (-29,000) \times 1 = 0 \\ Gamma &= 100,000 \times 0.047 + (-29,000) \times 0 = 4,700 \\ Vega &= 100,000 \times 0.099 + (-29,000) \times 0 = 9,900 \\ Theta &= 100,000 \times -.044 + (-29,000) \times 0 = -4,400 \end{aligned}$$

Exercise (cont'd)

- Tomorrow, XYZ stock opens at 95. Here is the new set of option prices and Greek letters.

	Market Price	delta	gamma	vega	theta
XYZ Stock	95	1.0	0	0	0
XYZ Call 105 strike, 1 month	0.30	0.10	0.025	.047	-.021
XYZ Put 95 strike, 1 month	3.35	-0.46	0.044	.108	-.052

4. If you liquidate right now, what would the profit or loss on the hedged position be?
5. If you don't liquidate, what stock trade will you need to do to become delta neutral again?

Exercise (cont'd)

	Market Price	delta	gamma	vega	theta
XYZ Stock	95	1.0	0	0	0
XYZ Call 105 strike, 1 month	0.30	0.10	0.025	.047	-.021
XYZ Put 95 strike, 1 month	3.35	-0.46	0.044	.108	-.052

4. If you unwind at the new prices your profit is:

$$\begin{aligned} Calls &= 100,000 \times (0.30 - 1.25) = -95,000 \\ Stocks &= -29,000 \times (95 - 100) = +145,000 \\ Total &= +50,000 \end{aligned}$$

5. If you wanted to rehedge, with the new delta, you should only be short

$$100,000 \times 0.10 = 10,000.$$

You have to buy back 19,000 of the shares you shorted.

Who cares about the Greeks?

- They are very important for market makers (MM).
- When a MM trades an option, s/he immediately trades stocks to cover delta risk.
  - MM is not betting on direction, but volatility.
- MM has a portfolio of different options, strikes, maturities and constantly monitoring the overall Delta, Gamma, Vega, and Theta portfolio risk.

Protected Principal Note

- Remember that this is an investment strategy where investors do not lose any of principal (initial investment) and sometimes earn additional profits. Also called "capital protected note".
- Investment banks often offer such securities, and hedge the short position with options or dynamic trading strategies.

Protected Principal Note: Example

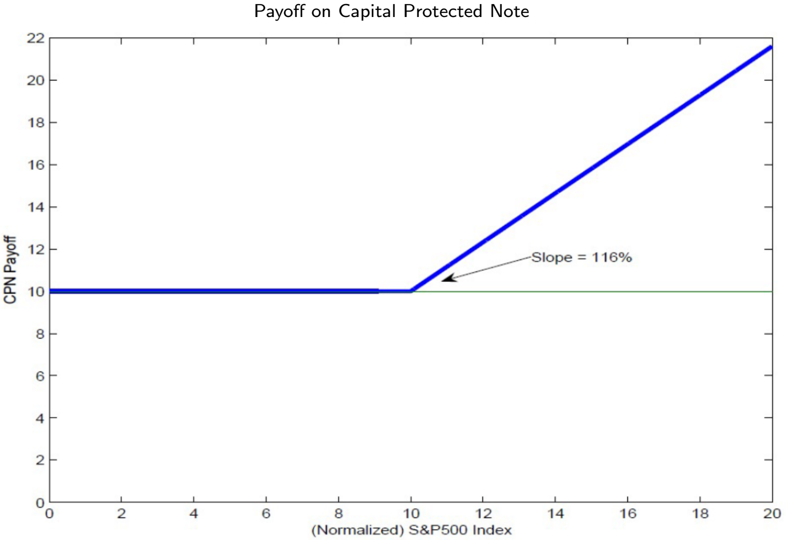
- On Feb 22, 2008, you as an MM **sold** a Capital Protected Note with:
  - Maturity: February 20, 2015
  - Issue price: \$10
  - Principal: \$10
  - Interest: 0%
  - Principal protection: 100%
  - Payoff at maturity = principal + Supplemental Redemption Amount (SRA) if positive

$$SRA = \$10 \times 116\% \times \frac{\text{Final Index Value} - \text{Initial Index Value}}{\text{Initial Index Value}}$$

  - Index is S&P 500 normalized to have Initial Index Value = \$10

- You want to protect your short position against increases in the stock price index.

Protected Principal Note: Example (cont'd)



Protected Principal Note: Example (cont'd)

- The payoff on the note can be decomposed into:
  - A zero coupon bond with principal \$10 and maturity $T = 7$.
  - 1.16 at-the-money call options on the normalized S&P 500 with maturity $T = 7$.
    - The reference index is normalized so that $S_0 = \beta \times S\&P500 = \$10$
    - On 2/28/08, $S\&P500 = 1353.1 \rightarrow \beta = 10/1353.1$

- Other data on 2/28/08
  - Interest rate, $r = 3.23\%$ (continuously compounded)
  - Dividend yield on S&P 500, $q = 2\%$
  - Forecast of market volatility over the 7 years, $\sigma = 15\%$

- The value of the security using BSM for dividend-paying stock is:

$$e^{-rT}(\$10) + (1.16)Call(S_0, K, r, \delta, \sigma, T)$$
$$= \$7.9764 + (1.16)\$1.7 = \$9.9483$$

- Investors give up interest on principal in exchange for a call option.

Protected Principal Note: Example (cont'd)

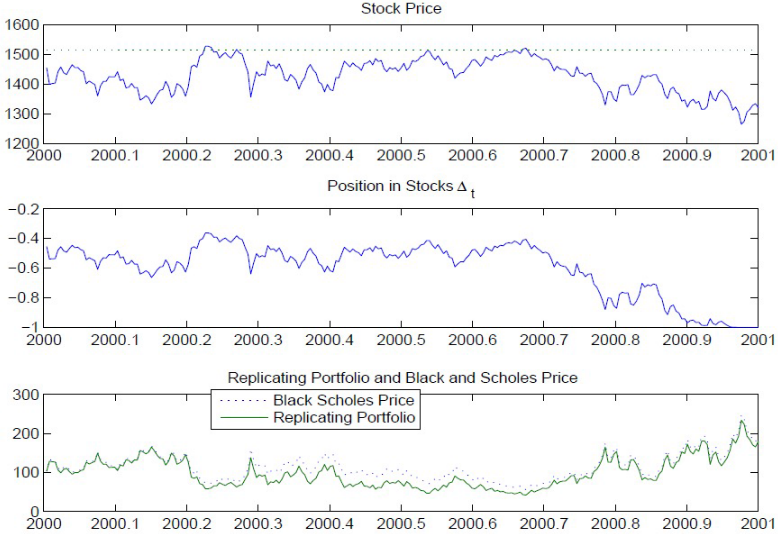
- At $t = 0$, the MM has a short position in the Capital Protected Note
- Hedge with an offsetting long position:
  - Buy a zero coupon bond for \$7.9764 to hedge the bond component.
  - Buy 1.16 units of the replicating portfolio for the embedded call option.
- Setting up the replicating portfolio for each call:
  - We can calculate the call's $\Delta = e^{-qT}N(d_1) = 0.5747$.
  - Then the bond position $= Call(S_0, K, r, \delta, \sigma, T) - \Delta \times S_0 = 1.7 - 0.5747 \times 10 = -4.047$
  - In sum, for each call option, invest $0.5747 \times \$10 = \$5.747$ in the S&P 500 and borrow \$4.047
- Value of replicating portfolio $= \$5.747 - \$4.047 = 1.7$
- Multiply both positions by 1.16 to scale up to the replicating portfolio for the Capital Protected Bond

Dynamic Delta Hedging

- Theoretically we need to frequently rebalance the portfolio as the Δ changes.
  - It will change with the stock price.
  - It will also change the passage of time, and any changes in r and σ.
- Recalculate Δ and new value of call.
- Adjust holdings of stocks and bonds in replicating portfolio to match new option value.
- The effectiveness of dynamic hedging depends on:
  - Frequency of rebalancing
  - Stability and accuracy of parameters (e.g., volatility)
  - Whether jumps in stock prices

How well does dynamic replication work in practice?

Replicating a put option on S&P 500 index, $T = 1, \sigma =$ std.dev in 1999.



Portfolio Insurance (LOR Method)

- In 1981, UC Berkeley Professors **Hayne Leland** and **Mark Rubinstein** partnered with **John O'Brien** to form **Leland, O'Brien, Rubinstein Associates (LOR)**.
    - **Business Idea:** Provide downside protection for portfolios using *dynamic replication* based on option pricing theory.
    - The goal was to replicate a **protective put** on a portfolio—without trading listed options— by dynamically adjusting stock and cash positions.
    - A fully invested equity portfolio (e.g., a pension fund) could obtain "insurance" guaranteeing a minimum floor value for the portfolio.
    - LOR did not sell insurance directly. Instead, they **advised clients on dynamic asset allocation**: when the market fell, increase short positions or reduce equity exposure; when the market rose, increase equity exposure—**mimicking option delta-hedging**.
    - Their product was attractive to **pension funds, endowments, and mutual funds** seeking capital preservation with equity exposure.
    - Adoption accelerated dramatically: although initially slow, demand grew rapidly during 1984–1986. By 1987, an estimated **$100 billion** in assets were using portfolio insurance strategies.

Portfolio Insurance: Example (cont'd)

- A portfolio is worth \$90 million. To protect against market downturns:
  1. Give the following data

$$S_0 = \$90, K = \$87, r = 0.09, q = 0.03, \sigma = 0.25, T = 0.5$$

$$d_1 = \frac{ln(90/87) + (0.09 - 0.03 + 0.25^2/2)0.5}{0.25\sqrt{0.5}} = 0.4499$$

  2. Create the put option synthetically, where the delta is $e^{-qT}(N(d_1) - 1) = -0.3215$.

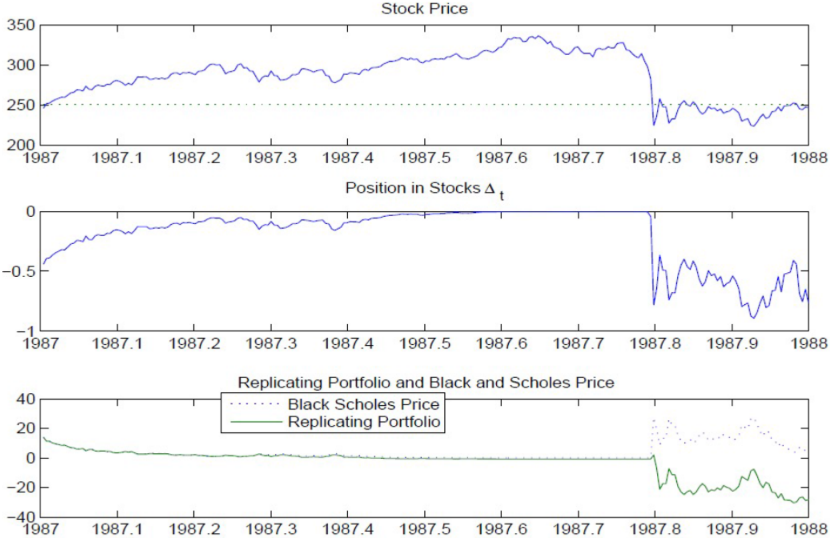
  - This shows that 32.15% of the portfolio should be sold initially and invested in risk-free assets.
  - If the value of the original portfolio reduces to \$88 million after 1 day, the delta of the required option changes to 0.3679 and a further 4.64% of the original portfolio should be sold and invested in risk-free assets.

Portfolio Insurance (cont'd)

- A variety of portfolio insurance structures emerged during the 1980s, all attempting to replicate some form of downside protection using dynamic trading.
- **The most important innovation was "perpetual" portfolio insurance.**
  - Traditional insurance products had a **fixed horizon** (e.g., 3 years). Protection lasted only for a predetermined period.
  - But institutional investors—especially **pension funds and endowments**—have **very long-term liabilities**. Short-term insurance is of limited use to investors managing decades-long obligations.
  - Perpetual insurance was designed to replicate a *perpetual American put* on the portfolio: the investor could "exercise" (lock in the floor value) at any time, indefinitely into the future.
  - This flexibility made perpetual insurance far more attractive, since the portfolio was always insured against large declines while still participating in the upside of the market.
- By the mid-1980s, perpetual insurance became the dominant form and drove explosive growth in demand— setting the stage for its massive influence on equity market dynamics.
- But then came the **1987 crash** . . .

How well does dynamic replication work in practice?

This example is for put options around the time of the 1987 market crash.



How Well Does Dynamic Replication Work in Practice?

- In theory, delta-hedging perfectly replicates an option *only* in a world where prices move continuously and trading can occur at infinitely high frequency (Black–Scholes assumptions).
- In practice, stock prices move in discrete steps and often experience **large, sudden jumps**. When this happens, the hedge cannot be adjusted quickly enough, and replication errors accumulate.
- Dynamic replication works reasonably well when:
  - price paths are smooth,
  - volatility changes gradually, and
  - markets are liquid enough to execute trades without moving prices.
- But when markets are volatile or discontinuous, the strategy breaks down:
  - hedging becomes **costly**,
  - slippage creates large tracking errors,
  - and replication fails to provide the promised downside protection.
- The 1987 crash is the classic example: the large one-day drop made continuous rebalancing impossible, revealing the limits of portfolio insurance strategies built on delta-hedging.

Delta–Gamma Hedging

- We have seen that pure **delta hedging** has important limitations:
  - The hedge must be rebalanced continuously, which is costly in the presence of transaction costs.
  - Large, discrete jumps in the stock price cause the hedge to fail because delta hedging assumes smooth (continuous) price paths.
- One way to improve hedge performance is **delta–gamma hedging**, which attempts to hedge not only the first-order sensitivity (delta) but also the second-order sensitivity (gamma).
- The idea is to add to the hedge portfolio a security with **positive gamma**—typically a short-term traded option. Positive gamma offsets the negative gamma of a long-dated short option position.

Delta–Gamma Hedging (cont'd

- Consider a portfolio:

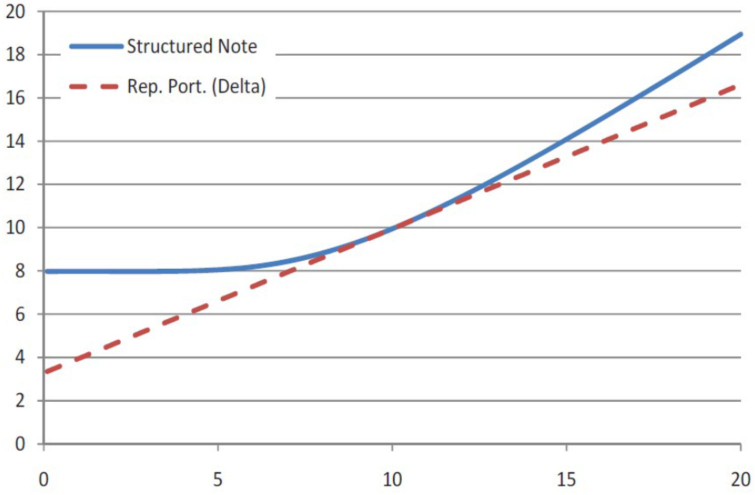
$$\Pi = -Call(S, T) + N \times S + N^C \times Call(S, T_1)$$

  where:
  - We are short a long-dated call (similar to the embedded call in a capital-protected note).
  - We hold N shares of stock.
  - We hold N^C units of a short-maturity call option with expiry $T_1 < T$.
- We choose N and N^C so that:

$$\frac{\partial \Pi}{\partial S} = 0 \quad \text{and} \quad \frac{\partial^2 \Pi}{\partial S^2} = 0.$$

- This means:
  - **Delta-neutrality**: small moves in S do not affect the portfolio value.
  - **Gamma-neutrality**: the portfolio's delta does not change when S moves.
- With gamma hedged, the portfolio becomes much more robust to large price moves, reducing the frequency and cost of rebalancing—even though perfect replication is still impossible in practice.

Delta hedging: Capital Protected Note



Delta–Gamma Hedging (cont'd)

- To eliminate both first- and second-order price risk, we impose:

$$\frac{\partial \Pi}{\partial S} = 0 \qquad \text{and} \qquad \frac{\partial^2 \Pi}{\partial S^2} = 0.$$

- Applying these to

$$\Pi = -C(S, T) + NS + N^C C_1(S, T_1),$$

we obtain:

$$\frac{\partial \Pi}{\partial S} = -\Delta(S, T) + N + N^C \Delta(S, T_1) = 0, \qquad \text{(Delta hedge)}$$

$$\frac{\partial^2 \Pi}{\partial S^2} = -\Gamma(S, T) + N^C \Gamma(S, T_1) = 0, \qquad \text{(Gamma hedge)}$$

- Solving the system yields:

$$N^C = \frac{\Gamma(S, T)}{\Gamma(S, T_1)}, \qquad N = \Delta(S, T) - N^C \Delta(S, T_1).$$

- Since the short-term option has positive gamma, we take a **long** position in it. This reduces the stock position relative to pure delta-hedging because we must hedge the delta of the short-term option as well.

Delta–Gamma Hedging (cont'd)

- Example: Hedge a long-dated call (the embedded option in a Capital Protected Note) using a one-year traded option.

$$C(S,T) = 1.7000, \quad \Gamma(S,T) = 0.0801, \quad \Delta(S,T) = 0.5747$$
$$C(S,T_1) = 0.6443, \quad \Gamma(S,T_1) = 0.2575, \quad \Delta(S,T_1) = 0.5512$$

- Compute hedge ratios:

$$N^C = \frac{\Gamma(S,T)}{\Gamma(S,T_1)} = \frac{0.0801}{0.2575} = 0.3113$$

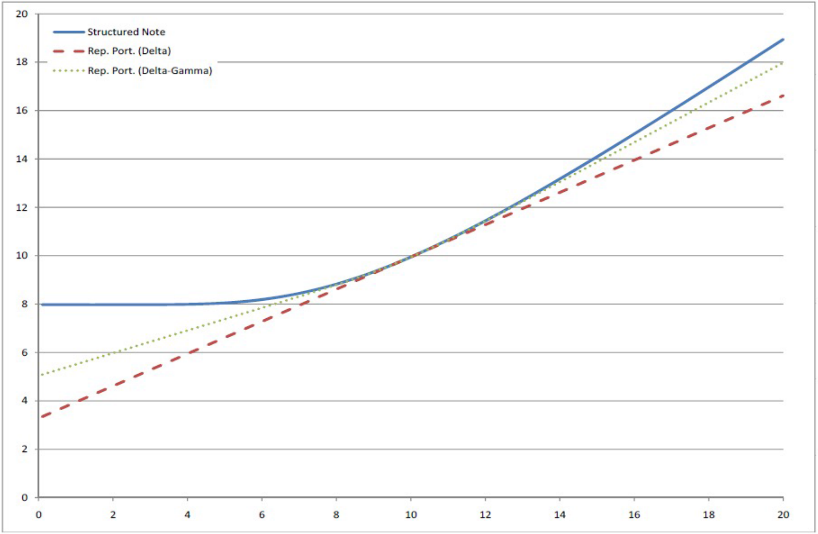
$$N = \Delta(S,T) - N^C \Delta(S,T_1) = 0.5747 - (0.3113)(0.5512) = 0.4031$$

- The bond position is then:

$$\text{Bonds} = C(S,T) - NS - N^C C(S,T_1) = -2.5315$$

- Interpretation:
  - We hold a smaller stock position (0.4031 instead of 0.5747 under delta-only hedging).
  - We hold a small long position in a short-dated call to neutralize gamma.
  - The negative bond position reflects the financing required to support the hedge.

Delta-gamma hedging (cont'd)



Delta-gamma hedging (cont'd)

- The Delta-Gamma hedging allows for larger swings in the stock price before calling for rebalancing.
- Less frequent rebalancing implies lower transaction costs.
  - But we have more transaction costs from rebalancing T_1-dated options.
  - We need to use very liquid, exchange-traded options to minimize transaction costs.
- Additional benefit is that large sudden changes in stock prices are now better hedged.

Why Gamma Hedges Reduce Rebalancing Frequency

- A pure **delta hedge** removes only first-order price sensitivity:

$$\frac{\partial \Pi}{\partial S} = 0.$$

  But as soon as S moves, delta changes:

$$\Delta \to \Delta + \Gamma \, \Delta S.$$

- A portfolio with **large gamma** experiences large changes in delta even for small moves in S:

$$|\Delta_{\text{new}} - \Delta_{\text{old}}| = |\Gamma \Delta S|.$$

  Hence, delta hedges break down quickly $\to$ frequent rebalancing required.
- By adding a traded option with **offsetting gamma**, we create:

$$\Gamma_{\Pi} \approx 0.$$

- With gamma neutralized:
  - Delta becomes almost constant in a neighborhood of S.
  - Small price movements no longer force immediate rebalancing.
  - Hedge remains effective over a wider range of underlying prices.

Why Delta–Gamma Hedging Still Fails in Practice

- Even though delta–gamma hedging improves replication, it still relies on key assumptions that often fail in real markets.
- **Volatility shocks**
  - Gamma hedging does not address volatility risk (vega).
  - When implied volatility changes, option values shift dramatically.
- **Liquidity constraints**
  - Adjusting stock or option positions may move the market price.
  - In stressed markets, liquidity disappears and hedging becomes impossible.
- **Model risk**
  - Hedging relies on Black–Scholes Greeks.
  - If volatility smile, jumps, or stochastic volatility are present, the Greeks are wrong.
- Bottom line: **Delta–gamma hedging reduces risk but can never fully eliminate it in real markets.**

Vega Hedging

- Consider again the portfolio

$$\Pi = -C(S,T) + NS + N^C C_1(S,T_1),$$

  where we are short a long-dated call $C(S,T)$, long N shares, and long N^C units of a traded short-dated call $C_1(S,T_1)$.
- Previously, we used C_1 to hedge gamma. But options also carry **vega risk**:

$$\nu = \frac{\partial C}{\partial \sigma},$$

  so we can also consider the sensitivity of Π to volatility.
- To make the portfolio vega-neutral:

$$\frac{\partial \Pi}{\partial \sigma} = -\nu(S,T) + N^C \nu(S,T_1) = 0.$$

Vega-hedging (cont'd)

- Solving for the hedge ratio:

$$N^C = \frac{\nu(S, T)}{\nu(S, T_1)}.$$

- Interpretation:
  - If the long-dated option has larger vega than the short-dated option (typical), we must take a larger position in C_1.
  - The stock position N does not affect vega (stock has zero vega).
  - Vega neutrality eliminates exposure to implied-volatility shocks.

Vega–Gamma Hedging

- To hedge both **gamma** and **vega** simultaneously, one traded option is not enough. We need at least two options with different maturities (or strikes).

$$\Pi = -C(S, T) + NS + N_1^C C_1(S, T_1) + N_2^C C_2(S, T_2).$$

- The hedging conditions are:

$$\text{Vega neutrality:} \qquad \frac{\partial \Pi}{\partial \sigma} = -\nu + N_1^C \nu_1 + N_2^C \nu_2 = 0,$$

$$\text{Gamma neutrality:} \qquad \frac{\partial^2 \Pi}{\partial S^2} = -\Gamma + N_1^C \Gamma_1 + N_2^C \Gamma_2 = 0.$$

Vega-Gamma hedging (cont'd)

- Solving the two-equation system:

$$N_2^C = \frac{\nu\,\Gamma_1 - \nu_1\,\Gamma}{\nu_2\,\Gamma_1 - \nu_1\,\Gamma_2}, \qquad N_1^C = \frac{\nu - N_2^C \nu_2}{\nu_1} = \frac{\Gamma - N_2^C \Gamma_2}{\Gamma_1}.$$

- Economic intuition:
  - Using two options allows us to cancel both curvature risk (Γ) and volatility risk (ν).
  - Options with different maturities have different gamma/vega ratios, giving us two degrees of freedom.
  - After choosing N_1^C and N_2^C, the stock position N is set to hedge delta.

Example

- Consider a portfolio that is delta neutral, with a gamma of $-5,000$ and a vega of $-8,000$.
- The options shown in the following table can be traded.

	Delta	*Gamma*	*Vega*
Portfolio	0	–5000	–8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

- Unlike the formula we derived where we hedge short call, here assume that we try to hedge long calls.

Example (cont'd)

- To make the portfolio gamma and vega neutral, both Option 1 and Option 2 can be used. If w_1 and w_2 are the quantities of Option 1 and Option 2 that are added to the portfolio, we require that

$$-5,000 + 0.5w_1 + 0.8w_2 = 0$$
$$-8,000 + 2.0w_1 + 1.2w_2 = 0$$

- The solution to these equations is $w_1 = 400, w_2 = 6,000$.
- The portfolio can therefore be made gamma and vega neutral by including 400 of Option 1 and $6,000$ of Option 2.
- The delta of the portfolio, after the addition of the positions in the two traded options, is $400 \times 0.6 + 6,000 \times 0.5 = 3,240$. Hence, $3,240$ units of the asset would have to be sold to maintain delta neutrality. This doesn't affect Gamma and Vega.

Mispricing When Greeks Are Wrong (Model Risk)

- Greek-based hedging relies on **model-derived sensitivities**:

$$\Delta_{\text{model}}, \quad \Gamma_{\text{model}}, \quad \nu_{\text{model}}, \ldots$$

- But the Greeks are only as good as the model behind them (e.g., Black–Scholes, local volatility, stochastic volatility).
- When the model is wrong, the Greeks are wrong:

$$\Delta_{\text{true}} \neq \Delta_{\text{model}}, \quad \Gamma_{\text{true}} \neq \Gamma_{\text{model}}, \ldots$$

- Sources of model risk:
  - Volatility smile/skew not captured by Black–Scholes.
  - Stochastic volatility, jumps, or fat tails.
  - Incorrect interest rates, dividend assumptions, or correlations.
  - Calibration errors (market data noise, sparse data).
- When Greeks are mismeasured:
  - Delta hedges drift away → persistent P&L leaks.
  - Gamma hedges fail to offset curvature → large convexity errors.
  - Vega hedges miss volatility shocks → P&L explosions on event days.
- A seemingly small Greek error leads to: Wrong hedge ratios, systematic under/over-hedging, drift in portfolio value, potential large losses during stress.

General Approach to Risk Management

- **Risk exposures add up linearly.** For any portfolio, the Greek exposures are the sum of the Greeks of its component instruments:

$$\Delta_\Pi = \sum_i \Delta_i, \quad \Gamma_\Pi = \sum_i \Gamma_i, \quad \nu_\Pi = \sum_i \nu_i, \ldots$$

  This linearity is what makes Greek-based hedging tractable.
- **At least one hedging instrument is needed per type of risk.**
  - To hedge two independent risks (e.g., delta and gamma), we need at least *two* traded instruments that react differently to those risks.
  - One hedging instrument $\Rightarrow$ can eliminate only one dimension of risk.
- **Not every instrument can hedge every Greek.**
  - A bond has zero delta, zero gamma, zero vega $\rightarrow$ cannot hedge market or volatility risk.
  - A stock has delta but zero gamma and zero vega $\rightarrow$ hedges only delta.
  - Two European options with the same strike and maturity have identical gamma and vega $\rightarrow$ cannot be used to hedge gamma and vega separately.
- Effective hedging requires selecting instruments whose Greek exposures span the risks you want to eliminate. This is a *dimensionality* problem $\rightarrow$ you need enough instruments and they must be sufficiently different.

General Approach to Risk Management (cont'd)

- **When the number of hedge instruments exceeds the number of risks, the hedge is not unique.**
  - Many different combinations of hedge ratios can deliver $\Delta = 0, \quad \Gamma = 0, \quad \nu = 0, \ldots$.
  - This creates a *family* of hedges, all satisfying the constraints.
- **This flexibility allows optimization along other dimensions:**
  - **Minimize hedging cost:** Cheaper combination of options and stock to achieve the same Greek neutrality.
  - **Maximize expected profit:** If some options appear mispriced, overweight "cheap" ones and underweight "expensive" ones while still matching Greeks.
  - **Minimize future rebalancing:** Choose hedge ratios that make the portfolio less sensitive to drift in Greeks $\rightarrow$ reduces transaction costs.
  - **Minimize liquidity impact:** Use more liquid instruments where possible to reduce slippage and market impact.
- When there is more than one feasible hedge, traders choose the hedge that best balances cost, robustness, liquidity, and views on mispricing—while still neutralizing the targeted risks.

Appendix: LTCM and Model Risk in Hedginga

LTCM and Model Risk in Hedging

- **Long-Term Capital Management (LTCM)** was a hedge fund founded by John Meriwether with Nobel laureates Merton and Scholes (Black–Scholes co-author).
- Their strategy relied heavily on:
  - relative-value arbitrage,
  - convergence trades,
  - extremely leveraged hedges based on model-derived Greeks.
- **Primary vulnerability: Model Risk**
  - Pricing models assumed stable correlations and smooth markets.
  - Greeks (especially delta, vega, rho) were computed assuming normal markets.
  - They believed risks were hedged because the models said so.

LTCM and Model Risk in Hedging (cont'd)

- **What went wrong? (1998 crisis)**
  - Russia defaulted on its sovereign debt.
  - Global flight to quality $\rightarrow$ massive spread widening and volatility spikes.
  - Correlations broke down (moves became highly nonlinear).
  - Liquidity evaporated $\rightarrow$ impossible to rebalance hedges.
- **Outcome:**
  - Model-based hedges failed spectacularly.
  - Small Greek errors became huge when volatility and spreads deviated from model assumptions.
  - Portfolio suffered massive losses; Fed-organized bailout followed.
- **Lesson:** Accurate hedging requires not just computing Greeks, but ensuring the **model generating those Greeks is valid under stress**. Otherwise, hedging creates leverage to model error.

Appendix: Relation between Delta, Gamma, and Theta

Relation Between Delta, Gamma, and Theta

- Black and Scholes derived a **partial differential equation (PDE)** that any option price $V(S, t)$ must satisfy in a world with continuous trading and no arbitrage.
- The Black–Scholes PDE is:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.$$

- Using Greek notation:

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV.$$

- For a **delta-neutral portfolio** ($\Delta = 0$):

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = rV.$$

Relation Between Delta, Gamma, and Theta (cont'd)

- **The PDE links time decay (Theta), curvature (Gamma), and the cost of carry.**
  - Θ (time decay) and Γ (curvature) must jointly generate the risk-free return rV.
  - High positive Theta (Gamma) typically requires strongly negative Gamma (Theta).
- **Economic meaning: the Theta–Gamma tradeoff**
  - Buying Gamma (e.g., long options) gives convexity but costs time decay (negative Theta).
  - Selling Gamma (e.g., short options) earns Theta but creates curvature risk (large negative Gamma).
  - In a delta-neutral portfolio, **Theta is effectively the cost or payoff of holding Gamma**.
- In a delta-neutral setting, Theta and Gamma are mechanically linked by the PDE. This is why traders often treat Theta as a proxy for Gamma exposure— you "earn" Theta only by taking on negative Gamma risk.

Appendix: Other Greeks

Other Higher-Order Greeks

- Volga $= \frac{\partial^2 V}{\partial \sigma^2}$: Change in Vega due to volatility
- Vanna $= \frac{\partial^2 V}{\partial \sigma \partial S}$: Change in Vega due to price
- Charm $= \frac{\partial^2 V}{\partial S \partial t}$: Change in Delta due to time passing