Black-Scholes-Merton Model

BUSS386. Futures and Options

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Lecture Outline

- Black-Scholes-Merton Model
  - Log-Normal Property of Stock Prices
  - Derivation
  - Iterpretation

BSM Model

Binomial Model vs. Black–Scholes–Merton (BSM) Model

- **Binomial model:** assumes the underlying asset price moves in discrete time-steps (up or down at each step).
- **BSM model:** built on continuous-time dynamics, modelling the asset price as evolving continuously.
- Although their approaches differ, they are closely connected: as the binomial time-steps shrink toward zero, the discrete model *converges* to the BSM formula for European-style options.
- When to use which:
  - Use the binomial model for flexibility (e.g., American options, early exercise, variable volatility).
  - Use the BSM model when assumptions (continuous trading, no early exercise, constant volatility) are reasonable and a closed-form solution is desired.

Black-Scholes-Merton Model

- The prices of European call and put options on non-dividend-paying stock are

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and $N(x)$ is the cumulative probability distribution function for a standard normal random variable.

BSM Model – Distribution of Future Stock Price

- A core assumption of the Black–Scholes–Merton model (BSM) is that the underlying stock price follows a log-normal distribution, i.e.

$$\ln S_T \sim N(m, s^2)$$

  where $N(m, s^2)$ denotes a normal distribution with mean m and variance s^2.[1]

- We can also derive this log-normal result via the discrete-time binomial model:
  - As the time-step size tends to zero and the number of steps tends to infinity, the binomial distribution of stock-price paths converges to the continuous GBM model and hence to log-normal terminal distribution.

- Next, we will prove the log-normality of S_T.

[1] Equivalently, S_T is log-normally distributed, which ensures $S_T > 0$ and aligns with modelling via Geometric Brownian Motion (GBM):

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

and thus

$$\ln S_T = \ln S_0 + \left(\mu - \tfrac{1}{2}\sigma^2\right) T + \sigma W_T.$$

Log-Normal Property of Stock Prices – Setup

- Consider a binomial tree for the stock price with n steps each of length $\Delta t = \dfrac{T}{n}$.
- At each step the stock either moves up by factor u or down by factor d.
- If there are j upward moves and $n - j$ downward moves, then at expiry

$$S_T(j) = S_0\, u^j\, d^{\,n-j}.$$

(This sets the discrete-time framework from which we will pass to a continuous-time limit.)

Distribution of Terminal Moves

- What will happen if n becomes infinitely large? (This would be equivalent to making each step infinitesimally small).

- To see this, let's increase the number of steps n.

Distribution of j when $n = 10$ and $p = 0.5$

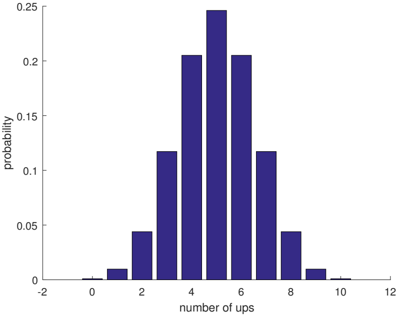
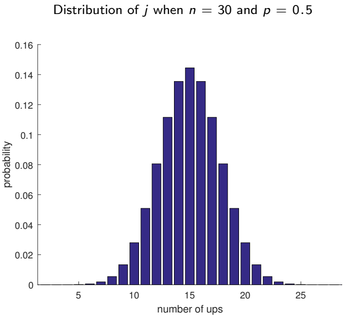
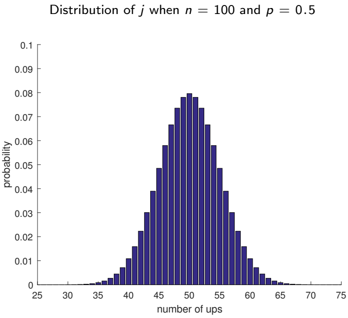


Illustration of Convergence





- As n grows large, the bar/histogram of j becomes more like a smooth bell curve. $\rightarrow$ The Central Limit Theorem

- Hence, as n approaches infinity, the number of upward movement will be normally distributed

$$j \sim N(np, \sqrt{np(1-p)}).$$

From Terminal Moves to Log Stock Price

- We now know the distribution of j. Next, let's find the distribution of $S_T(j)$.
- Using $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$,

$$
\begin{aligned}
S_T(j) &= S_0 u^j d^{n-j} \\
&= S_0 e^{(\sigma\sqrt{\Delta t})j} e^{(-\sigma\sqrt{\Delta t})(n-j)} \\
&= S_0 e^{(2\sigma\sqrt{\Delta t})j - n\sigma\sqrt{\Delta t}}
\end{aligned}
$$

- The log of stock price is

$$\ln S_T(j) = \ln S_0 + (2\sigma\sqrt{\Delta t})j - n\sigma\sqrt{\Delta t}$$

- As j is normally distributed, $\ln S_T$ is also normally distributed. Hence, $S_T(j)$ is log-normally distributed.

Mean and Variance of $\ln S_T$ (in limit)

- To further identify the distribution, let's find the mean and the standard deviation of $\ln S_T$.

- The mean of $\ln S_T$ is

$$\begin{aligned} E(\ln S_T) &= \ln S_0 + 2\sigma\sqrt{\Delta t}E(j) - n\sigma\sqrt{\Delta t} \\ &= \ln S_0 + 2\sigma\sqrt{\Delta t}(np) - n\sigma\sqrt{\Delta t} \end{aligned}$$

Mean and Variance of $\ln S_T$ (in limit)

- To proceed, we use $p = \frac{e^{r\Delta t} - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$ (i.e., risk-neutral probability). Here we use the Tylor series of e^x and also the fact $\Delta t \to 0$ as $n \to \infty$.[2]

$$
\begin{aligned}
p &\approx \frac{1 + r\Delta t - (1 - \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2)}{(1 + \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2) - (1 - \sigma\sqrt{\Delta t} + \sigma^2\Delta t/2)} \\
&= \left(\frac{1}{2} + \frac{(r - \sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right)
\end{aligned}
$$

- Plugging this p into $E(\ln S_T)$ in the previous page, the mean becomes

$$
E(\ln S_T) = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T
$$

where we use the fact $\Delta t = \frac{T}{n}$.

[2] $e^{r\Delta t} \approx 1 + r\Delta t + \frac{1}{2}r^2\Delta t^2$ (set Δt as x), $e^{-\sigma\sqrt{\Delta t}} \approx 1 - \sigma\sqrt{\Delta t} + \frac{1}{2}\sigma^2\Delta t$ (set $\sqrt{\Delta t}$ as x).

Mean and Variance of $\ln S_T$ (in limit)

- The standard deviation of $\ln S_T$ is

$$\begin{aligned}\text{Std.Dev.}(\ln S_T) &= 2\sigma\sqrt{\Delta t} \times \sqrt{np(1-p)} \\ &= 2\sigma\sqrt{Tp(1-p)}.\end{aligned}$$

- Next, let's simplify the standard deviation. We find

$$\begin{aligned}p(1-p) &= \left(\frac{1}{2} + \frac{(r-\sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right)\left(\frac{1}{2} - \frac{(r-\sigma^2/2)\sqrt{\Delta t}}{2\sigma}\right) \\ &= \frac{1}{4} - \frac{\left(r-\sigma^2/2\right)^2}{4\sigma^2}\Delta t \approx \frac{1}{4}\end{aligned}$$

- Thus, the standard deviation of $\ln S_T$ becomes

$$\text{Std.Dev.}(\ln S_T) = \sigma\sqrt{T}.$$

Log-Normal Property of Stock Prices

- Combining the mean and the standard deviation, we conclude

$$\ln S_T \sim \phi\left(\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma\sqrt{T}\right)$$

  in the risk-neutral world.

Log-Normal Property of Stock Prices - Real probability

- Consider the real world where investors require the return α per annum on stock. Then, we can use the real probability p^* instead of p.

$$p^* = \frac{e^{\alpha \Delta t} - d}{u - d}$$

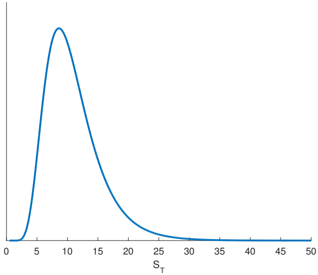
- Following the same logic as in the risk-neutral world, the real world distribution of stock price is

$$\ln S_T \sim \phi\left(\ln S_0 + \left(\alpha - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right)$$

Log-Normal Property of Stock Prices - Example

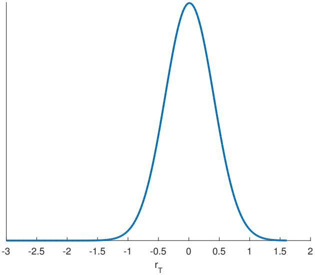
- Suppose that $S_0 = 10$, $r = 0.09$, $\sigma = 0.4$, and $T = 1$. Below are the probability density functions of S_T and r_T $(= \ln(S_T/S_0))$.

S_T



log-normal distribution

r_T



normal distribution

Probability of Option Exercise

- Using the distribution of future stock price under the risk-neutral measure, we can determine the probability of option exercise.
- Consider a European call with strike price K and expiration date T.
- What is the probability of option exercise,

$$\text{Prob}\left(S_T \geq K\right)$$

when S_T is log-normally distributed?

Probability of Option Exercise

- The probability is ...

$$
\begin{aligned}
&\text{Prob}\left(S_T \geq K\right) = \text{Prob}\left(\ln S_T \geq \ln K\right) \\
=&\text{Prob}\left(\frac{\ln S_T - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \geq \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&1 - \text{Prob}\left(\underbrace{\frac{\ln S_T - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}}_{\sim\phi(0,1)} < \frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&1 - N\left(\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \\
=&N\left(\frac{-\ln K + \ln S_0 + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right) \equiv N(d_2)
\end{aligned}
$$

where $\phi(0,1)$ is a standard normal random variable, and $N(x)$ is the cumulative distribution function of the standard normal.

Next

- Using the log-normal distribution of stock price, we can calculate the expected payoff of an option. This will lead us to the Black-Scholes-Merton formula.

- The exercise probability, $N(d_2)$, will be a part of the BSM result.

Math Review

Derivation of the BSM formula - Math Review

- In the derivation of the BSM formula, we need to compute the expected value of a function of random variable.
- This requires the understanding of a normal random variable and its probability density function.
- In addition, the calculation requires us to change variable in integration. This technique will be reviewed in the next slide.

Math Review - Normal Distribution

- Recall that to define $N(x)$, we consider a standard normal random variable Z.
- For a certain value x, $N(x)$ is the probability that Z is lower than or equal to x.

$$N(x) \equiv \text{Prob}(Z \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz.$$

- Graphically, $N(x)$ is the shadowed area in the below figure.

Figure 14.3 Shaded area represents $N(x)$.

- In Excel, we can use the function "norm.s.dist(x, TRUE)" to compute $N(x)$.

Math Review - Change of variable in integration

- Suppose that we integrate function $f(y)$ with respect to y:

$$\int f(y)dy.$$

- In addition, y is a function of another variable x, $y = g(x)$.

- Then, we can rewrite the above integration with respect to x

$$\int f(y)dy = \int f(g(x))g'(x)dx.$$

- Intuitively, we change dy to $g'(x)dx$ based on the derivative

$$\frac{dy}{dx} = g'(x)$$

Math Review - Change of variable in integration

e.g. Y is a normal random variable with mean m and the standard deviation w. $f(Y)$ is a function of the variable. Then, the expectation of $f(Y)$ is

$$E[f(Y)] = \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(y-m)^2}{2w^2}} dy$$

- Consider a new variable $z = \frac{y-m}{w}$. Then, $y = m + wz$ and $(dy) = w(dz)$. We can rewrite the above integration in terms of z:

$$\begin{aligned} \int_{-\infty}^{\infty} f(y) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(y-m)^2}{2w^2}} dy &= \int_{-\infty}^{\infty} f(m+wz) \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{z^2}{2}} w(dz) \\ &= \int_{-\infty}^{\infty} f(m+wz) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \end{aligned}$$

Derivation of Black-Scholes-Merton Model

Black-Scholes-Merton Model - Assumptions

- The derivation of BSM option price is based on following assumptions.
  - The stock price follows a log-normal distribution.
  - The risk-free rate, r, is constant and the same for all maturities.
  - There are no dividends during the life of the derivative.
  - There are no transaction costs or taxes.
  - There are no arbitrage opportunities.

Black-Scholes-Merton Model - Derivation

- Using the present-value approach, the call price is

$$c_0 = e^{-rT} E\left[\max(S_T - K, 0)\right].$$

when we compute the expected payoff under the risk-neutral probability $\Rightarrow$ Risk-neutral valuation

- Utilizing the log-normal distribution of S_T, we can compute the expected option payoff. Then, by discounting as above, we obtain the option price.

Black-Scholes-Merton Model - Derivation

- First, let's calculate $E\left[\max(S_T - K, 0)\right]$

- Note that S_T is log-normally distributed in the risk-neutral world.

$$\ln S_T \sim \phi\left(\underbrace{\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T}_{\equiv m}, \underbrace{\sigma\sqrt{T}}_{\equiv w}\right)$$

- To simplify the notation, let V denote $\ln S_T$. So, $V \sim \phi(m, w)$. Then, the probability density function of V is

$$g(V) = \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(V-m)^2}{2w^2}}$$

- Let's use $g(V)$ to compute the expected payoff of the call.

Black-Scholes-Merton Model - Derivation

- The expected payoff is

$$
\begin{aligned}
E\left[\max(S_T - K, 0)\right] &= E\left[\max(e^V - K, 0)\right] \\
&= \int_{-\infty}^{\infty} \max(e^V - K, 0) g(V) dV \\
&= \int_{-\infty}^{\ln K} \underbrace{\max(e^V - K, 0)}_{=0} g(V) dV + \int_{\ln K}^{\infty} \underbrace{\max(e^V - K, 0)}_{=e^V - K} g(V) dV \\
&= \int_{\ln K}^{\infty} (e^V - K) g(V) dV \\
&= \underbrace{\int_{\ln K}^{\infty} e^V \cdot g(V) dV}_{\equiv \mathbb{A}} - \underbrace{\int_{\ln K}^{\infty} K \cdot g(V) dV}_{\equiv \mathbb{B}}
\end{aligned}
$$

- Let's calculate $\mathbb{A}$ and $\mathbb{B}$ separately and combine later.

Black-Scholes-Merton Model - Derivation

- Let's find $\mathbb{B}$ first.

$$
\begin{aligned}
\mathbb{B} &= \int_{\ln K}^{\infty} K \cdot g(V) dV \\
&= K \int_{\ln K}^{\infty} g(V) dV \\
&= K \cdot \text{Prob}\left(V \geq \ln K\right) \\
&= K \cdot \text{Prob}\left(\underbrace{e^{V}}_{=S_T} \geq \underbrace{e^{\ln K}}_{=K}\right) \\
&= K \cdot N(d_2)
\end{aligned}
$$

where $d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}$.

Black-Scholes-Merton Model - Derivation

- Next, let's find $\mathbb{A}$.

$$\mathbb{A} = \int_{\ln K}^{\infty} e^V \cdot g(V) dV = \int_{\ln K}^{\infty} e^V \cdot \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{(V-m)^2}{2w^2}} dV$$

- To simplify the calculation, define a new variable $Q = \frac{V-m}{w}$. Then, $V = m + wQ$, and $(dV) = w(dQ)$ in the change of variable in the integration.

$$\begin{aligned}
\mathbb{A} &= \int_{\frac{\ln K - m}{w}}^{\infty} e^{m+wQ} \frac{1}{\sqrt{2\pi w^2}} e^{-\frac{Q^2}{2}} w \times dQ \\
&= \int_{\frac{\ln K - m}{w}}^{\infty} e^{m+wQ} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}} dQ \\
&= \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2}+wQ+m} dQ \\
&= \ldots
\end{aligned}$$

Black-Scholes-Merton Model - Derivation

$$
\begin{aligned}
\mathbb{A} &= \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2} + wQ - \frac{w^2}{2} + \frac{w^2}{2} + m} dQ \\
&= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Q^2}{2} + wQ - \frac{w^2}{2}} dQ \\
&= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(Q-w)^2}{2}} dQ
\end{aligned}
$$

- To simplify, define a new variable $Y = Q - w$. Then, $Q = Y + w$ and $(dQ) = (dY)$ in the change of variable in the integration.

$$
\begin{aligned}
\mathbb{A} &= e^{m + \frac{w^2}{2}} \int_{\frac{\ln K - m - w^2}{w}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{Y^2}{2}} dY \\
&= e^{m + \frac{w^2}{2}} \times \text{Prob}\left(Y \geq \frac{\ln K - m - w^2}{w} \right) \\
&= \ldots
\end{aligned}
$$

Black-Scholes-Merton Model - Derivation

$$\begin{aligned}
\mathbb{A} &= e^{m+\frac{w^2}{2}} \times \left[1 - \text{Prob}\left(Y < \frac{\ln K - m - w^2}{w}\right)\right] \\
&= e^{m+\frac{w^2}{2}} \times \left[1 - N\left(\frac{\ln K - m - w^2}{w}\right)\right] \\
&= e^{m+\frac{w^2}{2}} \times N\left(\frac{-\ln K + m + w^2}{w}\right)
\end{aligned}$$

Black-Scholes-Merton Model - Derivation

- In $\mathbb{A}$,

$$m + \frac{w^2}{2} = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T + \frac{\sigma^2 T}{2} = \ln S_0 + rT$$

$$\frac{-\ln K + m + w^2}{w} = \frac{\ln S_0 - \ln K + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

- Thus,

$$\begin{aligned}\mathbb{A} &= e^{m+\frac{w^2}{2}} \times N\left(\frac{-\ln K + m + w^2}{w}\right) \\ &= S_0 e^{rT} \times N\left(\underbrace{\frac{\ln S_0 - \ln K + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}}_{\equiv d_1}\right) \\ &= S_0 e^{rT} \times N(d_1)\end{aligned}$$

Black-Scholes-Merton Model - Derivation

- Now, let's combine $\mathbb{A}$ and $\mathbb{B}$.

$$\begin{aligned} E\left[\max(S_T - K, 0)\right] &= \mathbb{A} - \mathbb{B} \\ &= S_0 e^{rT} \times N(d_1) - K \times N(d_2) \end{aligned}$$

- The current price of the call is

$$\begin{aligned} c_0 &= e^{-rT} E\left[\max(S_T - K, 0)\right] \\ &= e^{-rT}\left[S_0 e^{rT} \times N(d_1) - K \times N(d_2)\right] \\ &= S_0 N(d_1) - K e^{-rT} N(d_2) \end{aligned}$$

Black-Scholes-Merton Model - Derivation

- Once the call option is obtained, we can easily drive the put price using the put-call parity.

$$
\begin{aligned}
p_0 &= c_0 + Ke^{-rT} - S_0 \\
&= S_0 N(d_1) - Ke^{-rT} N(d_2) + Ke^{-rT} - S_0 \\
&= -S_0 \left[1 - N(d_1)\right] + Ke^{-rT} \left[1 - N(d_2)\right] \\
&= -S_0 N(-d_1) + Ke^{-rT} N(-d_2)
\end{aligned}
$$

Black-Scholes-Merton Model

- The BSM model provides an analytic form that determines the option price as a function of the followings:
  - Current stock price S_0
  - Strike price K
  - Time to expiration T
  - Risk-free interest rate r
  - Volatility of underlying asset σ
- Through the BSM model, we can find the option price by simply inputting numbers into the option-pricing formula.

Black-Scholes-Merton Model - Result

- The prices of European call and put options on non-dividend-paying stock are

$$c_0 = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$p_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

and $N(x)$ is the cumulative probability distribution function for a standard normal random variable.

Black-Scholes-Merton Model - Example

Q. There is a 6-month European call option on a stock whose current price is \$42. The strike price is \$40, and the risk-free interest rate is 10% per annum. The stock volatility is 20% per annum. What is the price of the option?

Answer:

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln(42/40) + (0.1 + 0.2^2/2)(0.5)}{0.2\sqrt{0.5}} = 0.7693$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.6278$$

$$\begin{aligned} c &= S_0 N(d_1) - Ke^{-rT} N(d_2) \\ &= 42 \times N(0.7693) - 40e^{-0.1\times 0.5} \times N(0.6278) \\ &= 42 \times \text{norm.s.dist}(0.7693, \text{TRUE}) - 40e^{-0.1\times 0.5} \times \text{norm.s.dist}(0.6278, \text{TRUE}) \\ &= \$4.759. \end{aligned}$$

Black-Scholes-Merton Model - Example

- What if we use the binomial model for the previous question?

- Let's start with 10-step binomial model and increases the number of steps.

number of steps	option price
10	4.800
20	4.768
50	4.762
⋮	⋮
500	4.759
BSM price	4.759

- As the number of steps increases, the binomial price converges to the BSM price.

Black-Scholes-Merton Model – Example

Q. A European put option on a non-dividend-paying stock:

$$S_0 = \$60, \quad K = \$65, \quad T = 1 \text{ year}, \quad r = 5\% \text{ p.a.}, \ \sigma = 30\% \text{ p.a.}$$

What is the theoretical price of this put option under the BSM model?

Black-Scholes-Merton Model - Another Example

Q. Consider a derivative on a stock with the time to expiration T and the following payoff:

$$\begin{cases} 0 & \text{if } S_T < K_1 \\ K_1 & \text{if } K_1 \leq S_T < K_2 \\ 0 & \text{if } K_2 \leq S_T \end{cases}$$

where $K_2 > K_1$. What is the present value of the derivative? Provide an analytic expression of the price using $N(\cdot)$, the cumulative probability distribution function of a standard normal random variable.

Black-Scholes-Merton Model - Another Example

Answer: Let V denote $\ln S_T$. Then, V is normally distributed, i.e., $V \sim \phi(m, w)$. Let $g(V)$ denote the probability density function of V. To find the present value of the derivative, we first compute the expected option payoff:

$$\begin{aligned}
E\left[\text{Payoff}\right] &= \int_{-\infty}^{\infty} \text{Payoff} \cdot g(V)dV \\
&= \int_{-\infty}^{\ln K_1} \text{Payoff} \cdot g(V)dV + \int_{\ln K_1}^{\ln K_2} \text{Payoff} \cdot g(V)dV \\
&\qquad + \int_{\ln K_2}^{\infty} \text{Payoff} \cdot g(V)dV \\
&= \int_{-\infty}^{\ln K_1} 0 \cdot g(V)dV + \int_{\ln K_1}^{\ln K_2} K_1 \cdot g(V)dV + \int_{\ln K_2}^{\infty} 0 \cdot g(V)dV \\
&= K_1 \int_{\ln K_1}^{\ln K_2} g(V)dV \\
&= K_1 \cdot \text{Prob}\left(\ln K_1 \leq V \leq \ln K_2\right) \\
&= K_1 \cdot \text{Prob}\left(K_1 \leq S_T \leq K_2\right) \\
&= K_1 \cdot \left[\text{Prob}\left(K_1 \leq S_T\right) - \text{Prob}\left(K_2 \leq S_T\right)\right]
\end{aligned}$$

Black-Scholes-Merton Model - Another Example

Answer (cont'd):

$$= K_1 \cdot \left[N\left(\frac{\ln(S_0/K_1) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln(S_0/K_2) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right) \right].$$

Next, multiplying by the discount factor, we obtain the present value as follows:

$$f_0 = e^{-rT} K_1 \cdot \left[N\left(\frac{\ln(S_0/K_1) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right) - N\left(\frac{\ln(S_0/K_2) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} \right) \right].$$

BSM Formula: Interpretation

- Under the Black–Scholes–Merton model, a call option can be viewed as being replicated by a portfolio of the underlying stock and a risk-free bond.

- In particular:

$$\Delta_c = \frac{\partial C}{\partial S} = N(d_1) > 0,$$

meaning that $N(d_1)$ is the number of shares of stock held in the replicating portfolio for the call.

$$\Delta_p = \frac{\partial P}{\partial S} = -\,N(-d_1) < 0,$$

meaning for a put the equivalent position is short stock.

- The term $K\,e^{-rT}\,N(d_2)$ represents the present-value of the amount borrowed (or short-bond position) in the replicating portfolio for a call.

- Hence the call price is simply the cost of the replicating portfolio at time 0:

$$c_0 \;=\; \Delta_c\,S_0 \;-\; B \;=\; S_0\,N(d_1) \;-\; K\,e^{-rT}\,N(d_2).$$

Extending the BSM model

The BSM for dividend payout

- Suppose the underlying pays continuous dividend q .
  - Dividend should, for the purposes of option valuation, be defined as the reduction in the stock price.

- Replace the stock price S in the formula by Se^{-qT} [3]

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

, where $d_1 = \frac{\ln(S_0/K)+(r-q+\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. (called Merton model)

- Delta $= e^{-qT} N(d_1)$

- Put-Call parity: $p + S_0 e^{-qT} = c + Ke^{-rT}$
  - Given the price of puts and calls, we can solve this for the "implied dividend yield q".

[3] For sketch of proof, go to the slide, "The BSM for dividend payout: Derivation".

Options on Futures – Call and Put Payoffs

- **Call on a futures contract:**
  - Right to enter a long futures position at the strike price K.
  - On exercise, the payoff $= \max(F_T - K, 0)$, where F_T is the futures price at expiry.
- **Put on a futures contract:**
  - Right to enter a short futures position at strike K.
  - Payoff $= \max(K - F_T, 0)$.
- These payoffs are analogous to vanilla options on assets, but the underlying is a futures contract instead of owning the asset.

Option on Futures – Example

- On August 15, a trader holds a September futures-call option on copper.
  - Strike price $K = 320$ cents per pound.
  - One futures contract represents 25,000 pounds of copper.
  - The current (most recent settlement) futures price for September delivery is $F = 330$ cents/pound.
  - The quoted "spot" (closing) price just before exercise is 331 cents/pound.

- If the option is exercised today, then:
  - The payoff from the option part is

$$25,000 \times (330 - 320) = 250,000 \text{ cents} = \$2{,}500$$

  - Immediately after exercise the trader receives the long futures contract (i.e., obligation to buy 25,000 pounds at 330).
  - If the trader decides to close out the futures position immediately (i.e., offset it), there is an additional gain equal to

$$25,000 \times (331 - 330) = 25,000 \text{ cents} = \$250.$$

  - Therefore the total payoff on exercise $= \$2{,}500 + \$250 = \$2{,}750$, which equals

$$25{,}000 \times (F - K) = 25{,}000 \times (331 - 320) \text{ cents} = \$2{,}750.$$

Options on Futures – Key Features & Advantages

- Advantages of futures-based options
  - Futures contracts often trade on highly liquid exchanges, making the underlying option more liquid and easier to hedge.
  - Exercise of a futures option does not require physical delivery of the underlying asset — instead the holder enters a futures position and may immediately offset it.
  - The option and the futures contract typically trade on the same exchange, which can reduce margin/clearing costs and simplify operational logistics.

- Equivalence for European style: If the option expires when the futures contract matures (i.e., $F_T = S_T$), then a European futures option is equivalent to a European spot option.

- Market scope
  - Common underlying futures for these options include: agricultural commodities (e.g., wheat, corn), energy (e.g., crude oil, natural gas), precious metals (e.g., gold, silver), interest-rate futures, and volatility indexes (e.g., VIX futures).
  - Many listed futures options are American style, allowing exercise at any time before expiry, especially in commodity markets.

Options on Futures: Black-76 (BSM Variant)

- The underlying is a futures contract, so S in the equation is the futures price, call it F.
  - Remember $F_0 = S_0 e^{rT}$. As time passes, e^{rT} shrinks at the rate of r like dividend yield q. (assume Futures = Forward here).

- Replace the stock price S in the formula by the discounted value of the futures price F: Fe^{-rT}

$$c = Fe^{-rT}N(d_1) - Ke^{-rT}N(d_2) = e^{-rT}[FN(d_1) - KN(d_2)]$$

, where $d_1 = \frac{\ln(F/K)+(\sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

- Delta $= e^{-rT}N(d_1)$

- Put-Call parity: $p + Fe^{-rT} = c + Ke^{-rT}$

Options on Futures – The Black '76 Model

- The model originates from Fischer Black's 1976 paper, "The Pricing of Commodity Contracts", where he extended the Black–Scholes–Merton model to options written on futures/forwards.

- Key features of the model:
  - You avoid separate modelling of convenience yields, storage costs or asset-income flows, because these are embedded in the forward/futures price.
  - The underlying is a forward/futures price (rather than owning the physical asset), which simplifies the replication and hedging.
    - Provided interest rates are deterministic (and hence forwards $\approx$ futures), this substitution is valid.
  - The forward/futures price is assumed to follow a log-normal distribution, similar to the BSM setup.
  - The model has wide applicability beyond commodity futures—e.g., interest-rate caps/floors, bond options, swaptions.

- Caveats:
  - The formula produces a European-style option value. For American-style options on futures, one must use alternative methods (e.g., binomial tree, finite difference).
  - If interest rates or cost-of-carry vary stochastically, the equivalence between forwards & futures may break, and more complex models are needed.

The BSM for currency option

- The price of the underlying is the exchange rate (in \$ per unit of FX). The underlying pays interest at the foreign riskless rate, so set $q = r_F$. The riskless rate r is the domestic rate (Garman-Kohlhagen Model).

- Replace the stock price S in the formula by $Se^{-r_F T}$

$$c = S_0 e^{-r_F T} N(d_1) - Ke^{-rT} N(d_2)$$

, where $d_1 = \frac{\ln(S_0/K) + (r - r_F + \sigma^2/2)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

- Delta $= e^{-r_F T} N(d_1)$

- Put-Call parity: $p + S_0 e^{-r_F T} = c + Ke^{-rT}$

- Using the Black's model: $c = e^{-rT}[FN(d_1) - KN(d_2)]$, where F is the futures price on currency.

Alternative Derivation I

Review

- This derivation is also based on the Binomal Tree model in the risk-neutral world.
  - The final stock price: $S_0 u^j d^{n-j}$.
  - The payoff from a European call option: $\max(S_0 u^j d^{n-j} - K, 0)$
  - The probability of j upward and $n-j$ downward steps: $\frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}$
  - The expected payoff: $\sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$
  - The option value: $c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K, 0)$

Review: Binomial Tree Derivation

- We begin from the multi-step binomial model in the risk-neutral world.
  - Final stock price after n steps:

$$S_T(j) = S_0\, u^j\, d^{\,n-j}$$

  - Payoff of a European call:

$$\max\big(S_0\, u^j\, d^{\,n-j} - K,\ 0\big)$$

  - Probability of exactly j upward moves (and $n-j$ downward):

$$\Pr(j) = \frac{n!}{j!\,(n-j)!}\, p^j\, (1-p)^{\,n-j}$$

  - Expected (risk-neutral) payoff:

$$\sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\, p^j\, (1-p)^{\,n-j}\, \max\big(S_0\, u^j\, d^{\,n-j} - K,\ 0\big)$$

  - Present value (call price):

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{j!\,(n-j)!}\, p^j\, (1-p)^{\,n-j}\, \max\big(S_0\, u^j\, d^{\,n-j} - K,\ 0\big)$$

Alternative Formulation of Call Price

Payoff positive if $S_0\, u^j\, d^{\,n-j} > K \quad \Rightarrow \quad \ln\!\left(\frac{S_0}{K}\right) > -j\, \ln(u) \;-\; (n-j)\, \ln(d)$

With $u = e^{\sigma\sqrt{T/n}},\ d = e^{-\sigma\sqrt{T/n}}$

$$\Rightarrow \quad \ln\!\left(\tfrac{S_0}{K}\right) > n\,\sigma\sqrt{\tfrac{T}{n}} \;+\; 2\,j\,\sigma\sqrt{\tfrac{T}{n}}$$

$$\Rightarrow \quad j > \frac{n}{2} \;-\; \frac{\ln(S_0/K)}{2\,\sigma\sqrt{T/n}}$$

Thus: $c = e^{-rT} \displaystyle\sum_{j>\alpha} \frac{n!}{j!(n-j)!}\; p^j\,(1-p)^{n-j}\; \max\!\left(S_0\, u^j\, d^{\,n-j} - K, 0\right),$

where $\alpha = \dfrac{n}{2} - \dfrac{\ln(S_0/K)}{2\,\sigma\sqrt{T/n}}$

Write $c = e^{-rT}\,(S_0\, U_1 - K\, U_2),$

with $U_1 = \displaystyle\sum_{j>\alpha} \frac{n!}{j!(n-j)!}\; p^j\,(1-p)^{n-j}\; u^j\, d^{\,n-j},$

$$U_2 = \sum_{j>\alpha} \frac{n!}{j!(n-j)!}\; p^j\,(1-p)^{n-j}$$

Increasing the Number of Steps: Convergence to BSM

As $n \to \infty$, $j \sim B(n,p) \longrightarrow \phi\big(np,\ \sqrt{np(1-p)}\big)$.

Since $U_2 = \Pr(j > \alpha)$, $\quad U_2 = \Pr\left(\frac{j - np}{\sqrt{np(1-p)}} > \frac{\alpha - np}{\sqrt{np(1-p)}}\right) = N\left(\frac{np - \alpha}{\sqrt{np(1-p)}}\right)$

$$\Rightarrow U_2 = N\left(\frac{\ln(S_0/K)}{2\,\sigma\,\sqrt{T\,p(1-p)}} + \frac{\sqrt{n}(p - \frac{1}{2})}{\sqrt{p(1-p)}}\right)$$

Recall $p = \dfrac{e^{r\,T/n} - e^{-\sigma\,\sqrt{T/n}}}{e^{\sigma\,\sqrt{T/n}} - e^{-\sigma\,\sqrt{T/n}}}$,

and by Taylor expansion: $e^{r\,T/n} \approx 1 + r\,(T/n)$, $e^{\pm\sigma\,\sqrt{T/n}} \approx 1 \pm \sigma\sqrt{\frac{T}{n}} + \frac{1}{2}\sigma^2\,(T/n)$.

Hence $p(1-p) \to \frac{1}{4}$ and $\sqrt{n}(p - \frac{1}{2}) \to \dfrac{(r - \frac{1}{2}\sigma^2)\sqrt{T}}{2\,\sigma}$.

$$\Rightarrow U_2 = N\left(\frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)\,T}{\sigma\,\sqrt{T}}\right)$$

Final Step: From Binomial to Black–Scholes

$$U_1 = \sum_{j>\alpha} \frac{n!}{j!(n-j)!} \,(u\,p)^j\,(d\,(1-p))^{\,n-j}.$$

Let $p^* = \dfrac{p\,u}{p\,u + (1-p)\,d}$, $\;1 - p^* = \dfrac{(1-p)\,d}{p\,u + (1-p)\,d}$.

$$\Rightarrow U_1 = (p\,u + (1-p)\,d)^{\,n} \sum_{j>\alpha} \frac{n!}{j!(n-j)!} \,(p^*)^j\,(1-p^*)^{\,n-j}.$$

Because $p\,u + (1-p)\,d = e^{rT}$, $\Rightarrow U_1 = e^{rT} \sum_{j>\alpha} \dfrac{n!}{j!(n-j)!} \,(p^*)^j\,(1-p^*)^{\,n-j}$.

So in the limit as $n \to \infty$, $U_1 = e^{rT}\; N\!\left(\dfrac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)\,T}{\sigma\sqrt{T}} \right)$.

$$\Rightarrow c = S_0\,N(d_1) - K\,e^{-rT}\,N(d_2).$$

Alternative Derivation II

Overview of Option-Pricing Derivations

- The previous binomial-tree derivation sets up a discrete framework:
  - The underlying asset and a risk-free bond are combined to build a portfolio that exactly replicates the option payoff at each node.
  - By enforcing no-arbitrage (the replicating portfolio must earn the risk-free rate), we derive the fair option price.
- In contrast, the Black–Scholes–Merton model (BSM) uses a continuous-time framework:
  - The option and the underlying asset are dynamically hedged to create a riskless position.
  - Since this hedged position must grow at the risk-free rate, we obtain a partial differential equation whose solution gives the option price.
- Key point: Although the approaches differ (discrete vs. continuous), both rely on the same principle of constructing a riskless arbitrage-free portfolio and enforcing that it returns the risk-free rate.

Underlying Assumptions of the BSM Model

- Options are European
- "Perfect" markets – no transactions costs, no taxes, no constraints on short selling with full use of the proceeds, no indivisibilities, etc.
- No limits on borrowing or lending at a known risk free rate of interest
- The price of the underlying asset follows a "lognormal diffusion" process
- The return volatility of the underlying asset is known
- No dividends or cash payouts from the underlying asset prior to option maturity

Asset Price Process in Continuous Time

- The model assumes the underlying asset price S_t follows a *geometric Brownian motion*:

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dz_t \quad \Longrightarrow \quad \frac{dS_t}{S_t} = \mu\, dt + \sigma\, dz_t.$$

- Explanation of components:
  - dS_t: instantaneous change in the price at time t.
  - μ: the drift (average continuously-compounded rate of return).
  - dt: an infinitesimal increment of time.
  - σ: volatility (annualised standard deviation of returns).
  - dz_t: increment of a standard Brownian motion (mean 0, variance dt).

- Key assumptions behind this model:
  - μ and σ are constant over time.
  - The process has independent increments (no memory, Markov property) and is continuous in time.
  - The asset can be traded continuously without transaction costs or liquidity constraints.

Key Definitions

- A process $\{z(t) : t \geq 0\}$ is called a Brownian motion (Wiener process) if:
  1. $z(0) = 0$.
  2. It has continuous paths and independent increments: for $0 \leq s < t$, the increment $z(t) - z(s)$ is independent of the past and distributed $N(0,\, t - s)$.
  3. Over a very small time interval Δt, one can think informally:

     $$dz_t \approx \epsilon \sqrt{\Delta t}, \quad \epsilon \sim N(0,1).$$

     For example, if $\Delta t = 0.01$ and $\epsilon = 1.5$, then $dz_t \approx 1.5 \times \sqrt{0.01} = 0.15$.

- A Generalised Wiener process is of the form:

  $$dS_t = \mu\, dt + \sigma\, dz_t,$$

  where μ and σ are constants.
  - Example: Suppose $\mu = 0.05$, $\sigma = 0.2$. Over a small $\Delta t = 0.25$ years, one might approximate:

    $$dS_t \approx 0.05 \times 0.25 + 0.2 \times dz_t.$$

    If $dW_t = 0.1$, then $dX_t \approx 0.0125 + 0.02 = 0.0325$.

Key Definitions (cont'd)

- An Itô process has the more general form:

$$dS_t = \mu(S_t, t)\, dt + \sigma(S_t, t)\, dz_t,$$

  where the drift and volatility can depend on the current state and time.

  - Example: Suppose an asset price satisfies:

  $$dS_t = \mu S_t\, dt + \sigma S_t\, dz_t$$

    with $\mu = 0.08$, $\sigma = 0.25$, $S_0 = 100$. Then S_t follows a geometric Brownian motion.

The Process for a Stock Price

- $dS_t = \mu dt$?
  - There is no uncertainty.
  - $S_t = \mu t$, i.e., stock price grows by μ. $\Rightarrow$ Not realistic!
- $dS_t = \mu dt + \sigma dz$?
  - There is uncertainty, dz.
  - But stock price can take a negative value!
- $dS_t / S_t = \mu dt + \sigma dz$
  - The most widely used model of stock price behavior.
  - For a risk-free asset, $\mu = r$ and $\sigma = 0$. Hence, $S_t = e^{rt}$.
  - Ito process, log-normal diffusion process, geometric Brownian motion

Examples – Part 1

- **Example 1: Arithmetic Brownian Motion (ABM)**

$$dX_t = \mu\, dt + \sigma\, dz_t$$

  - Here μ and σ are constants.
  - Suppose $\mu = 0.02$, $\sigma = 0.15$, and time horizon $T = 1$ year. If $X_0 = 100$, then the expected value is $E[X_T] = 100 + 0.02 \times 1 = 100.02$.
  - Variance is $\sigma^2 T = 0.15^2 \times 1 = 0.0225$. So the standard deviation is about $\sqrt{0.0225} \approx 0.15$.
  - This process can go negative; it models absolute changes rather than proportional changes.

- **Example 2: Geometric Brownian Motion (GBM)**

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dz_t$$

  - Suppose $\mu = 0.08$, $\sigma = 0.20$, and $S_0 = 50$. Then under the model, S_t remains strictly positive.
  - The log-return $\ln(S_t/S_0)$ is normal. This is the model assumed in the Black–Scholes–Merton model.
  - If we look at expected value: $E[S_t] = S_0 e^{\mu t} = 50e^{0.08\times 1} \approx 54.17$ (for $t = 1$ year) assuming no discounting.

Examples – Part 2

- **Example 3: Mean-Reverting Ornstein-Uhlenbeck (OU) Process**

$$dY_t = \kappa\big(\theta - Y_t\big)\,dt + \sigma\,dz_t$$

  - Let $\kappa = 1.5$, $\theta = 100$, $\sigma = 10$, starting value $Y_0 = 120$.
  - Interpretation: the process tends to revert toward long-term level $\theta = 100$ with speed κ.
  - Over time the expected value moves: $E[Y_t] = \theta + (Y_0 - \theta)e^{-\kappa t} = 100 + 20\,e^{-1.5\,t}$. For $t = 1$: $100 + 20e^{-1.5} \approx 100 + 20 \times 0.223 = 104.46$.
  - Use case: modelling interest rates or commodity spreads which tend to bounce back toward an equilibrium.

- **Example 4: Geometric Mean-Reverting Process**

$$dS_t = \kappa\big(\theta - \ln S_t\big)S_t\,dt + \sigma\,S_t\,dz_t$$

  - Here the drift term drives $\ln S_t$ toward θ; volatility is proportional to S_t.
  - Suppose $\kappa = 0.8$, $\theta = \ln(80)$, $\sigma = 0.25$, $S_0 = 60$.
  - The process tends to revert to an equilibrium level around $S \approx 80$. Useful in modelling commodity prices with proportionate volatility and mean reversion.

Examples – Part 3

- **Example 5: Cox–Ingersoll–Ross (CIR) Interest Rate Process**

$$dr_t = \kappa\big(\theta - r_t\big)\,dt + \sigma\sqrt{r_t}\,dz_t$$

  - A canonical model for short-term interest rates (ensuring $r_t \geq 0$).
  - Let $\kappa = 0.5$, $\theta = 0.04$, $\sigma = 0.1$, $r_0 = 0.03$.
  - Over time the rate moves toward 0.04, and volatility is state-dependent: $\sqrt{r_t}$.
  - Use case: pricing interest rate derivatives under stochastic rate models.

Ito's Lemma

- Suppose x follows an Ito process: $dx = a(x,t)dt + b(x,t)dz$
- Kiyoshi Ito (1915-2008) shows that a function of x and t, $G(x,t)$ (twice continuously differentiable) follows another Ito process:

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b dz$$

  - Apply a Taylor series expansion on $G(x,t)$:

$$dG \approx \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} dx^2 + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} dt^2 + \frac{\partial^2 G}{\partial x \partial t} dxdt$$

    - $dx^2 \approx b^2 dz^2 = b^2 \epsilon^2 dt$[4]
    - $E(b^2 \epsilon^2 dt) = b^2 dt$ and $Var(\epsilon^2 dt) = 2dt^2 \approx 0$ ($\because Var(\epsilon^2) = E(\epsilon^4) - E(\epsilon^2)^2 = 3 - 1 = 2$).
    - Ignore higher order terms (e.g. $dt^{1.5}, dt^2$).

$$dG \approx \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt$$

  - Plug in $dx = a(x,t)dt + b(x,t)dz$.

[4] $dtdz = 0$ and $(dz)^2 = dt$

Ito's Lemma (cont'd)

- Why this matters for option pricing:
  - When we let $G =$ option value $V(S_t, t)$, and S_t follows a geometric Brownian motion, applying Itô's Lemma lets us derive the partial differential equation that leads to the Black–Scholes Equation.
  - Understanding this term is central to moving from discrete-time models (binomial) to continuous time derivations.

Applying Itô's Lemma to S_t

- Apply Ito's lemma on $dS_t = \mu S_t dt + \sigma S_t dz_t$

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S dz$$

- Now consier $G = \ln S_t$.

$$\frac{\partial G}{\partial S} = \frac{1}{S_t}, \quad \frac{\partial^2 G}{\partial S^2} = -\frac{1}{S_t^2}, \quad \frac{\partial G}{\partial t} = 0$$

- Therefore,

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz$$

- It follows a generalized Wiener process.
  - $G_T - G_0 = \ln S_T - \ln S_0 \sim \phi\left(\left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right)$
  - That is, $\ln S_T \sim \phi\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right)$

Key Consequences of Itô's Lemma for GBM

- Under this model $dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dz_t$:
  - Continuously-compounded return dS_t/S_t is normally distributed (infinitesimal time).
  - Future stock price S_T has a log-normal distribution—implying $S_T > 0$.
- The same Brownian increment dz_t drives both the asset and any smooth function of it—for example $\ln S_t$.
- The log-normal assumption of S_T underlies the analytic closed-form formula for European option prices.

Deriving the Black–Scholes PDE – Step 1

- Assume the underlying asset price follows

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dz_t.$$

- Let $V = V(S_t, t)$ be the price of a European call option (a function of the asset price and time).

- Applying Itô's Lemma gives:

$$dV = \left(\frac{\partial V}{\partial S}\, \mu S_t + \frac{\partial V}{\partial t} + \tfrac{1}{2}\, \frac{\partial^2 V}{\partial S^2}\, \sigma^2 S_t^2 \right) dt + \frac{\partial V}{\partial S}\, \sigma S_t\, dz_t.$$

- Long 1 unit of the call option and short $\frac{\partial V}{\partial S}$ number of shares. (Why?)

$$\Pi = V - \frac{\partial V}{\partial S}\, S_t$$

  and compute its differential:

$$d\Pi = dV - \frac{\partial V}{\partial S}\, dS_t = \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\, \frac{\partial^2 V}{\partial S^2}\, \sigma^2 S_t^2 \right) dt$$

  (The dz_t term cancels by design.)

Deriving the Black–Scholes PDE – Step 2

- Because this equation does not involve dz, the portfolio must be riskless during time dt. Therefore,

$$\Pi = e^{rdt}$$
$$d\Pi = r\Pi dt$$
$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right) dt = r\left(V - \frac{\partial V}{\partial S} S\right) dt$$
$$\Rightarrow \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV$$

- This is called the Black–Scholes–Merton differential equation.
- Solving the differential equation with the boundary conditions, e.g., $V = \max(S - K, 0)$ when $= T$, gives a formula for a European call option.
    - Unfortunately, no exact analytic formula for the value of an American put option on a non-dividend-paying stock has been produced.

NB There is no μ, the expected return!

The BSM Differential Equation

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 = rV$$

- If $V(S,T) = S_T$, i.e., the stock itself, $V(S,t) = S_t$
- If $V(S,T) = K$, i.e., constant, then $V(S,t) = Ke^{-r(T-t)}$
- If $V(S,T) = S_T - K$, i.e., forward, then $V(S,t) = S_t - Ke^{-r(T-t)}$
- Does $V(S,0) = S_0 N(d_1) - Ke^{-rT} N(d_2)$ satisfy the equation?
- The PDE above is so general that it can solve (mostly numerially) for V depending on the boundary conditions.

The Black–Scholes PDE – Verification of Special Cases

$$\frac{\partial V}{\partial t} + r\,S\,\frac{\partial V}{\partial S} + \tfrac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} = r\,V$$

- If $V(S,T) = S_T$, i.e., the underlying stock itself $\rightarrow V(S,t) = S_t$.
  - Then $\frac{\partial V}{\partial t} = 0$, $\frac{\partial V}{\partial S} = 1$, $\frac{\partial^2 V}{\partial S^2} = 0$.
  - Substituting: $0 + r\,S \cdot 1 + \frac{1}{2}\sigma^2 S^2 \cdot 0 = r\,S = r\,V$.
  - So the PDE holds.

- If $V(S,T) = K$, a constant payoff $\rightarrow V(S,t) = K\,e^{-r\,(T-t)}$.
  - Then $\frac{\partial V}{\partial S} = 0$, $\frac{\partial^2 V}{\partial S^2} = 0$, and $\frac{\partial V}{\partial t} = r\,K\,e^{-r\,(T-t)} = r\,V$.
  - Substituting: $r\,V + r\,S \cdot 0 + 0 = r\,V$.
  - The PDE is satisfied.

- If $V(S,T) = S_T - K$ (a forward payoff) $\rightarrow V(S,t) = S_t - K\,e^{-r\,(T-t)}$.
  - Then $\frac{\partial V}{\partial S} = 1$, $\frac{\partial^2 V}{\partial S^2} = 0$, $\frac{\partial V}{\partial t} = -r\,K\,e^{-r\,(T-t)}$.
  - Left side: $-r\,K\,e^{-r\,(T-t)} + r\,S \cdot 1 + 0 = r\,V$.
  - Again the PDE holds.

Verification that the Call Price Satisfies the PDE

- It also holds for a European option on a non-dividend-paying stock. It's more complicated to verify, though.
- The PDE is extremely general. What changes between contracts (stock, bond, forward, option) is the *terminal condition* (and any boundary conditions). Once you know the terminal condition, you pick the corresponding solution that satisfies the PDE. Refer to standard derivations.

Appendix 1: The BSM for dividend payout

BSM with Continuous Dividend Yield: Derivation (1)

- Suppose the stock pays a continuous dividend yield q. Then, during dt, the stockholder receives a dividend

$$dD = q\, S\, \frac{\partial V}{\partial S}\, dt.$$

- The change in the value of the hedged portfolio is the sum of the change in portfolio value and the dividend income:

$$dW_t = d\Pi + dD.$$

- Using Itô's Lemma and the hedge ratio $\frac{\partial V}{\partial S}$, we have:

$$dW_t = \left(-\frac{\partial V}{\partial t} - \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + qS\frac{\partial V}{\partial S}\right) dt.$$

- Since the portfolio is instantaneously riskless, it must earn the risk-free rate r:

$$dW_t = r\,\Pi\, dt = r\left(-V + S\frac{\partial V}{\partial S}\right) dt.$$

BSM with Continuous Dividend Yield: Derivation (2)

- Equating the two expressions for dW_t and rearranging gives:

$$\frac{\partial V}{\partial t} + (r-q)S\frac{\partial V}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV.$$

- This is the **Black–Scholes–Merton PDE with dividends.** The dividend yield q reduces the drift of the stock under the risk-neutral measure.

- The corresponding risk-neutral stock price process is:

$$dS = (r-q)S\,dt + \sigma S\,dz.$$

- For a European call, solving the PDE gives the **Black–Scholes formula with dividends:**

$$c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2),$$

where

$$d_{1,2} = \frac{\ln(S_0/K) + (r - q \pm \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

Appendix 2: From BSM PDE to BSM equation

Step 1: The Black–Scholes PDE

- We start with the partial differential equation (PDE) for the option value $V(S,t)$:

$$\frac{\partial V}{\partial t} + r\,S\,\frac{\partial V}{\partial S} + \tfrac{1}{2}\,\sigma^2\,S^2\,\frac{\partial^2 V}{\partial S^2} \;=\; r\,V.$$

- Here:
  - S = underlying stock price at time t.
  - r = risk-free interest rate (continuous).
  - σ = volatility of the stock's returns.
  - The terminal (boundary) condition is:

$$V(S,T) = \max(S-K,\,0),$$

    for a European call option with strike K and maturity T.
- This PDE comes from hedging + Itô's Lemma + no-arbitrage.

Step 2: Change of Variables

- Solving the PDE directly is hard, so we perform a change of variables to simplify it. Typical transformations include:
  - $\tau = T - t$ (time to maturity).
  - $x = \ln(S/K)$ (log-stock variable).
  - Introduce a new function $u(x, \tau) = e^{r\tau} V(S, t)$ so that the discount-term rV disappears.
- Under these changes, the PDE is transformed into a "heat equation" form (a simpler diffusion PDE), for which standard solutions are known.
- This step is therefore a mathematical trick to make the PDE solvable with known methods.

Step 3: Solve the Transformed PDE

- Once in the "heat-equation" form, one applies known solution methods (e.g., separation of variables, Green's functions) to find $u(x, \tau)$.
- Then we revert the change of variables:

$$V(S, t) = e^{-r(T-t)}\, u\big(\ln(S/K),\; T - t\big).$$

- The result is an expression involving the normal cumulative distribution function $N(\cdot)$.
- In returning to the original variables, we obtain the closed-form formula for a European call option:

$$C = S\, N(d_1) \;-\; K\, e^{-r(T-t)}\, N(d_2),$$

with

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\,\sigma^2)\,(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = d_1 - \sigma\sqrt{T - t}.$$

Step 4: Interpretation & Key Insights

- Notice that the expected stock return μ does not appear in the final formula — only the risk-free rate r and volatility σ.
- Why? Because of risk-neutral valuation: in a hedged portfolio the expected return of the underlying becomes irrelevant.
- The formula therefore = discounted expected payoff under the "risk-neutral measure".