Practice Problem Set: Solutions

BUSS386 Futures and Options

1 Replicating Portfolio

A stock price is currently \$40. It is known that at the end of one month it will be either \$42 or \$38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-month European call option with a strike price of \$39?

Solution: Consider a portfolio consisting of:

- -1: Call option
- $+\Delta$: Shares

If the stock price rises to \$42, the portfolio is worth $42\Delta - 3$. If the stock price falls to \$38, it is worth 38Δ. These are the same when

$$42\Delta - 3 = 38\Delta$$

or $\Delta = 0.75$. The value of the portfolio in one month is 28.5 for both stock prices. Its value today must be the present value of 28.5, or $28.5e^{-0.08 \cdot 0.08333} = 28.31$. This means that

$$-f + 40\Delta = 28.31$$

where f is the call price. Because $\Delta = 0.75$, the call price is $40 \times 0.75 - 28.31 = \$1.69$.

As an alternative approach, we can calculate the probability, p, of an up movement in a risk-neutral world. This must satisfy:

$$42p + 38(1-p) = 40e^{0.08 \cdot 0.08333}$$

so that

$$4p = 40e^{0.08 \cdot 0.08333} - 38$$

or $p = 0.5669$. The value of the option is then its expected payoff discounted at the risk-free rate:

$$[3 \times 0.5669 + 0 \times 0.4331]e^{-0.08 \cdot 0.08333} = 1.69$$

or \$1.69. This agrees with the previous calculation.

2 Binomial Trees: One-Step

A stock price is currently \$100. Over each of the next two six-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a one-year European call option with a strike price of \$100?

Solution: In this case $u = 1.10$, $d = 0.90$, $\Delta t = 0.5$, and $r = 0.08$, so that

$$p = \frac{e^{0.08 \cdot 0.5} - 0.90}{1.10 - 0.90} = 0.7041$$

The tree for stock price movements is shown in Figure below. We can work back from the end of the tree to the beginning, as indicated in the diagram, to give the value of the option as \$9.61. The option value can also be calculated directly:

$$\left[0.7041^2 \times 21 + 2 \times 0.7041 \times 0.2959 \times 0 + 0.2959^2 \times 0\right] e^{-2 \cdot (0.08 \cdot 0.5)} = 9.61$$

or \$9.61.

3 Binomial Trees: Call Option with Two-Step

A stock price is currently \$50. Over each of the next two three-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a six-month European call option with a strike price of \$51?

Solution: A tree describing the behavior of the stock price is shown in Figure below. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.05 \cdot 3/12} - 0.95}{1.06 - 0.95} = 0.5689$$

There is a payoff from the option of $56.18 - 51 = 5.18$ for the highest final node (which corresponds to two up moves), zero in all other cases. The value of the option is therefore

$$5.18 \times 0.5689^2 \times e^{-0.05 \cdot 6/12} = 1.635$$

This can also be calculated by working back through the tree. The value of the call option is the lower number at each node in the figure.

4 Binomial Trees: Put Option with Two-Step

For the situation considered in the previous problem, what is the value of a six-month European put option with a strike price of \$51? Verify that the European call and European put prices satisfy put-call parity. If the put option were American, would it ever be optimal to exercise it early at any of the nodes on the tree?

Solution: The tree for valuing the put option is shown in Figure below. We get a payoff of $51 - 50.35 = 0.65$ if the middle final node is reached and a payoff of $51 - 45.125 = 5.875$ if the lowest final node is reached. The value of the option is therefore

$$\left(0.65 \times 2 \times 0.5689 \times 0.4311 + 5.875 \times 0.4311^2\right) e^{-0.05 \cdot 6/12} = 1.376$$

This can also be calculated by working back through the tree as indicated in Figure S13.4. The value of the put plus the stock price is

$$1.376 + 50 = 51.376$$

The value of the call plus the present value of the strike price is

$$1.635 + 51e^{-0.05 \cdot 6/12} = 51.376$$

This verifies that put-call parity holds.
To test whether it is worth exercising the option early, we compare the value calculated for the option at each node with the payoff from immediate exercise. At node C, the payoff from immediate exercise is $51 - 47.5 = 3.5$. Because this is greater than 2.8664, the option should be exercised at this node. The option should not be exercised at either node A or node B.

5 Binomial Trees: Application

A stock price is currently \$25. It is known that at the end of two months it will be either \$23 or \$27. The risk-free interest rate is 10% per annum with continuous compounding. Suppose S_T is the stock price at the end of two months. What is the value of a derivative that pays off S_T^2 at this time?

Solution: At the end of two months, the value of the derivative will be either 529 (if the stock price is 23) or 729 (if the stock price is 27). Consider a portfolio consisting of:

- $+\Delta$: shares
- -1: derivative

The value of the portfolio is either $27\Delta - 729$ or $23\Delta - 529$ in two months. If

$$27\Delta - 729 = 23\Delta - 529$$

i.e., $\Delta = 50$, the value of the portfolio is certain to be 621. For this value of Δ, the portfolio is therefore riskless. The current value of the portfolio is:

$$50 \times 25 - f$$

where f is the value of the derivative. Since the portfolio must earn the risk-free rate of interest

$$(50 \times 25 - f)e^{0.10 \cdot 2/12} = 621$$

i.e., $f = 639.3$. The value of the option is therefore \$639.3.
This can also be calculated directly from equations (13.2) and (13.3). $u = 1.08$, $d = 0.92$, so that

$$p = \frac{e^{0.10 \cdot 2/12} - 0.92}{1.08 - 0.92} = 0.6050$$

and

$$f = e^{-0.10 \cdot 2/12}(0.6050 \times 729 + 0.3950 \times 529) = 639.3$$

6 Finding u and d for Currency Option

Calculate u, d, and p when a binomial tree is constructed to value an option on a foreign currency. The tree step size is one month, the domestic interest rate is 5% per annum, the foreign interest rate is 8% per annum, and the volatility is 12% per annum.

Solution: $p = \frac{a-d}{u-d}$, where $a = e^{(r-r_f)\Delta t}$.
In this case

$$\begin{aligned}
a &= e^{(0.05-0.08) \cdot 1/12} = 0.9975 \\
u &= e^{0.12\sqrt{1/12}} = 1.0352 \\
d &= 1/u = 0.9660 \\
p &= \frac{0.9975 - 0.9660}{1.0352 - 0.9660} = 0.4553
\end{aligned}$$

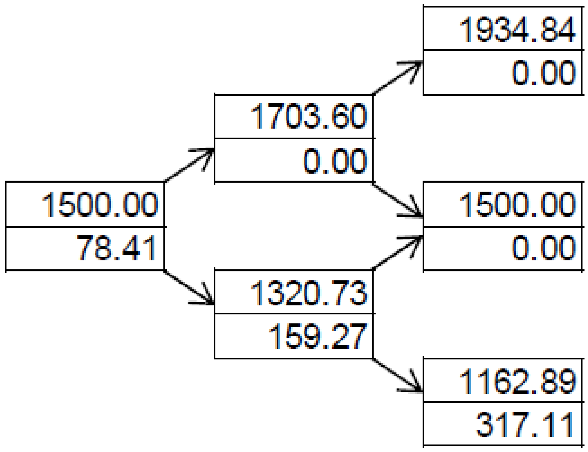
7 Finding u and d for Index Option

A stock index is currently 1,500. Its volatility is 18%. The risk-free rate is 4% per annum (continuously compounded) for all maturities and the dividend yield on the index is 2.5%. Calculate values for u, d, and p when a six-month time step is used. What is the value of a 12-month American put option with a strike price of 1,480 given by a two-step binomial tree.

Solution:

$$\begin{aligned}
u &= e^{0.18\sqrt{0.5}} = 1.1357 \\
d &= 1/u = 0.8805 \\
p &= \frac{e^{(0.04-0.025) \times 0.5} - 0.8805}{1.1357 - 0.8805} = 0.4977
\end{aligned}$$

The tree is shown below. The option is exercised at the lower node at the six-month point. It is worth 78.41.



8 Binomial Trees: European vs. American

A stock price is currently \$40. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 12% per annum with continuous compounding.

a What is the value of a six-month European put option with a strike price of \$42?

b What is the value of a six-month American put option with a strike price of \$42?

Solution:

a A tree describing the behavior of the stock price is shown below. The risk-neutral probability of an up move, p, is given by

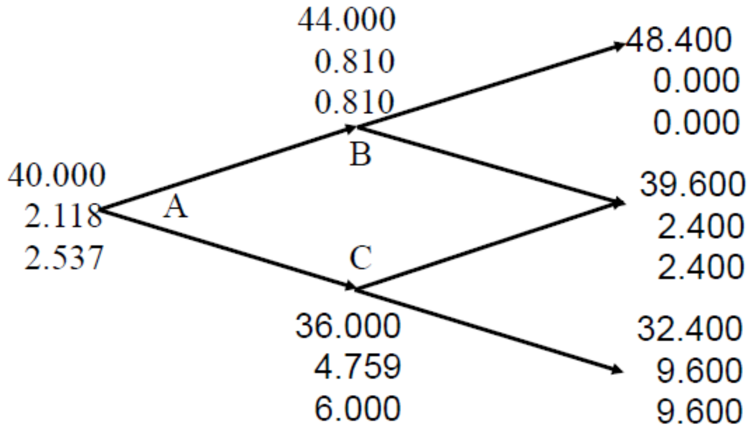
$$p = \frac{e^{0.12 \cdot 3/12} - 0.90}{1.1 - 0.9} = 0.6523$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$\left[2.4 \times 2 \times 0.6523 \times 0.3477 + 9.6 \times 0.3477^2\right] e^{-0.12 \cdot 6/12} = 2.118$$

The value of the European option is 2.118. This can also be calculated by working back through the tree in the figure. The second number at each node is the value of the European option.

b The value of the American option is shown as the third number at each node on the tree. It is 2.537. This is greater than the value of the European option because it is optimal to exercise early at node C.



Tree to Evaluate European and American Put Options: At each node, the upper number is the stock price, the next number is the European put price, and the final number is the American put price.

9 Binomial Trees: Application

A stock price is currently \$30. During each two-month period for the next four months it is expected to increase by 8% or reduce by 10%. The risk-free interest rate is 5%. Use a two-step tree to calculate the value of a derivative that pays off $[\max(30 - S_T, 0)]^2$ where S_T is the stock price in four months? If the derivative is American-style, should it be exercised early?

Solution: This type of option is known as a power option. The risk-neutral probability of an up move, p, is given by

$$p = \frac{e^{0.05 \cdot 2/12} - 0.9}{1.08 - 0.9} = 0.6020$$

Calculating the expected payoff and discounting, we obtain the value of the option as

$$\left[0.7056 \times 2 \times 0.6020 \times 0.3980 + 32.49 \times 0.3980^2\right] e^{-0.05 \times 4/12} = 5.393$$

The value of the European option is 5.393. This can also be calculated by working back through the tree. The option should not be exercised early if it is American.