Binomial Trees

BUSS386. Futures and Options

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Lecture Outline

- One-Step Binomial Tree
- Risk-Neutral Valuation vs. DCF
- Two-Step Binomial Tree
- N-Step Binomial Tree
- Advanced Topics in Binomial Models
    - American Options
    - Determination of u and d

Determining Option Prices

- We have characterized option prices using lower/upper bounds and the put-call parity. Still, we don't yet have a tool to determine the exact price of an option.
- To find the exact price of an option, we need a model describing how the underlying stock price will move in the future.
- Consider, again, a European call option.

$$\underbrace{e^{-r_{\text{call}}T}}_{\text{discounting factor}} \times \underbrace{E\left[\max(S_T - K, 0)\right]}_{\text{option payoff}}$$

- Biggest challenge for directly discounting future cash flows on options is difficulty of identifying the cost of capital
  - Technically that's because the implicit leverage in an option position is constantly changing over time, and the amount of leverage affects the discount rate

Determining Option Prices

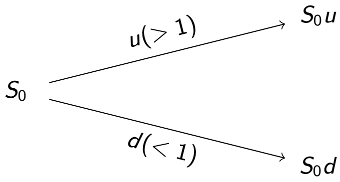
- A no-arbitrage approach, which can be implemented with binomial pricing, avoids the need to explicitly identify the relevant cost of capital

- Binomial trees incorporate the six main factors affecting the price of a stock option:
  - (1) current stock price; (2) strike price; (3) time to expiration; (4) volatility of the stock price; (5) risk-free interest rate; (6) expected dividends

Binomial Model - Setting

- Assumptions
  - Stock price follows a random walk.

    : In one time step, the stock price can move up or down by a certain amount (only two possible paths).



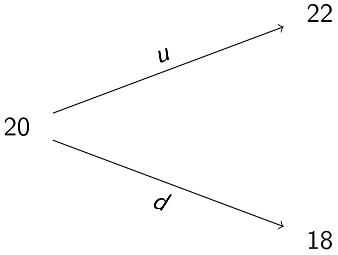
  - There is no arbitrage.

- This may look too simplistic to reflect the reality. Later, we will extend the model to allow multiple steps until the option expiration.

One-Step Binomial Model

e.g. A 3-month European call option has strike price \$21. The risk-free interest rate is 12% per annum.

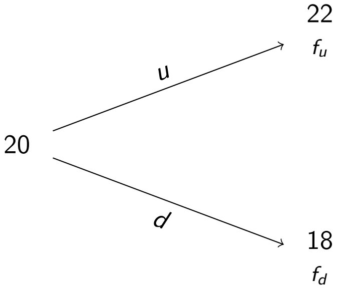
- Current stock price is \$20. The price can move either up to \$22 or down to \$18 during the life of the option.



- What is the price of the call option?

One-Step Binomial Model

1. To price the call, we first determine option payoffs at T:



- f_u(option value when stock price is up)$= \max(22 - K, 0) = 1$
- f_d(option value when stock price is down) $= \max(18 - K, 0) = 0$

One-Step Binomial Model

2. Next, we find a portfolio that replicates the option payoff in every case at T (using stock and bond):

   - Let x denote the number of shares and y the face value of the bond (in dollar) in the replicating portfolio. We want x and y such that

$$\begin{cases} 22x + y = 1 & \text{(at } u\text{)} \\ 18x + y = 0 & \text{(at } d\text{)} \end{cases}$$

   - Solving for the unknowns gives $x = 0.25$, $y = -4.5$. Thus, the replicating portfolio consists of buying 0.25 shares and selling a bond with the face value of -$4.5.

One-Step Binomial Model

3. The option price at time 0 should be equal to the price of the replicating portfolio. Otherwise, an arbitrage exists.

   - The price of the replicating portfolio is

   $$S_0 x + y e^{-rT} = 20 \times 0.25 - 4.5 e^{-0.12 \times 3/12} = 0.633$$

   - Hence, the option price is \$0.633.

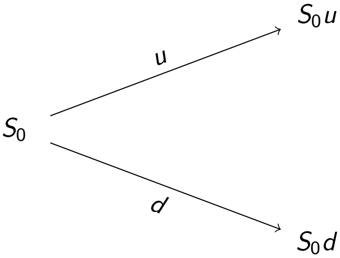
   - Note that we do not consider the probabilities of going up or down!

   - In fact, there probabilities are already reflected in the current stock price.

One-Step Binomial Model - General case

- A European call option has strike price K and expiration date T. The risk-free interest rate is r per annum.

- Current stock price is S_0. The price can move either up by u or down by d during the life of the option.



- Notice that we do not assign any probability for up/down movement.

One-Step Binomial Model - General case

1. Find option payoffs at T: $\begin{cases} f_u &= \max(S_0 u - K, 0) \\ f_d &= \max(S_0 d - K, 0) \end{cases}$

2. Find the replicating portfolio (x: number of shares, y: face value of bond).

   - We want to find x and y such that

$$\begin{cases} (S_0 u)x + y = f_u & \text{(at } u\text{)} \\ (S_0 d)x + y = f_d & \text{(at } d\text{)} \end{cases}$$

   - Solving for the unknowns, we obtain

$$x = \frac{f_u - f_d}{S_0 u - S_0 d}, \quad y = \frac{u f_d - d f_u}{u - d}$$

One-Step Binomial Model - General case

3. Calculate the present value (at time 0) of the replicating portfolio.

$$
\begin{aligned}
S_0 x + y e^{-rT} &= \frac{f_u - f_d}{u - d} + e^{-rT} \frac{u f_d - d f_u}{u - d} \\
&= e^{-rT} \left[\frac{e^{rT} f_u - e^{rT} f_d}{u - d} + \frac{u f_d - d f_u}{u - d} \right] \\
&= e^{-rT} \left[\frac{e^{rT} - d}{u - d} f_u + \frac{u - e^{rT}}{u - d} f_d \right] \\
&= e^{-rT} \left[p \times f_u + (1 - p) \times f_d \right]
\end{aligned}
$$

where $p = \frac{e^{rT} - d}{u - d}$.

One-Step Binomial Model

e.g. Go back to the previous example of the European call option with $K = 21$, and $T = 3$ months. The risk-free interest rate is 12% per annum. The current stock price is 20. The price can either move up to 22 or down to 18 during the life of the option.

- We priced the option using the replicating portfolio. Alternatively, we can use the option pricing formula. Here, $u = 22/20 = 1.1$ and $d = 18/20 = 0.9$.

- Then,

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 1/4} - 0.9}{1.1 - 0.9} = 0.652$$

- The price of the call option is

$$\begin{aligned} e^{-rT}\left[pf_u + (1-p)f_d\right] &= e^{-0.12 \times 1/4}\left[0.652 \times 1 + (1 - 0.652) \times 0\right] \\ &= \$0.633 \end{aligned}$$

Interpretation: Risk-Neutral Valuation

- The option price $e^{-rT}\left[pf_u + (1-p)f_d\right]$ is similar to the form we would obtain from DCF, when p is interpreted as a probability.

- However, the form is not exactly the same as the DCF.
  - Recall that in DCF, a riskier cash flows is discounted at a higher rate, say r_{call} (e.g. CAPM).
  - However, in the result of option price, the risky option payoff is discounted at the risk-free interest rate.

- Risk-Neutral Valuation
  - The discount rate in the option price is determined **as if** investors do not require a higher return for a riskier investment, that is, as if they are **risk-neutral**.

Risk-Neutral Valuation vs. DCF

- We call p the risk-neutral probability.
- p is different from the real probability we observe in data. To distinguish, let p^* denote the real probability of an increase in the stock price.
- Risk-neutral valuation vs. DCF

	Option Pricing (Risk-Neutral Valuation)	DCF
Discount rate	r	r_{call}
Probability	p	p^*
Option price	$e^{-rT}\left[pf_u + (1-p)f_d\right]$	$e^{-r_{\text{call}}T}\left[p^* f_u + (1-p^*)f_d\right]$

Pricing Options Using DCF

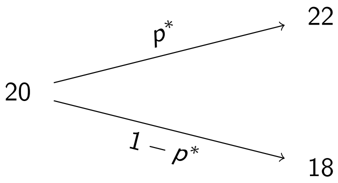
- Instead of risk-neutral valuation, we could price an option using the discounted cash flow (DCF) approach.
- To price an option using DCF, we need to find the real probability p^* and the discount rate r_{call} for the option.
- First, to find p^*, we suppose that risk-averse investors require the stock return to be α per annum in the real world.
- Then, we solve for p^* by setting α equal to the expected return in the binomial tree.

$$p^* = \frac{e^{\alpha T} - d}{u - d}$$

Why? $p^* S_0 u + (1 - p^*) S_0 d = S_0 e^{\alpha T}$

Pricing Options Using DCF

Q1. Revisit the previous example, where stock price in 3 months is given as below. In the real world, risk-averse investors require the return on stock to be 16% per annum. What is the real probability p^* of an increase in the stock price?



Pricing Options Using DCF

- In pricing using DCF, the next step is to find the discount rate r_{call}.
- To determine the discount rate, we use the fact that a portfolio of stock and bond replicates the call option in the binomial tree.
- Then, the required return on option is the weighted average of stock return (α) and bond return (r).
- The weight is determined by the fraction of stock and bond components in the portfolio.

$$\begin{cases} \text{weight on stock: } \frac{S_0 x}{S_0 x + y e^{-rT}} \\ \text{weight on bond: } \frac{y e^{-rT}}{S_0 x + y e^{-rT}} \end{cases}$$

Pricing Options Using DCF

Q2. In the previous tree, consider a 3-month call option with the strike price of 21. The risk-free rate is 12% per annum. What is the discount rate r_{call} for the call in DCF?

Pricing Options Using DCF

Q2. In the previous tree, consider a 3-month call option with the strike price of 21. The risk-free rate is 12% per annum. What is the discount rate r_{call} for the call in DCF?

Answer: r_{call} is the weighted average of stock return and bond return, where the weight is the fraction of an asset in the entire portfolio value.

$$
\begin{aligned}
e^{r_{\text{call}} \times 3/12} &= \frac{S_0 x}{S_0 x + y e^{-rT}} \times e^{\alpha \times 3/12} + \frac{y e^{-rT}}{S_0 x + y e^{-rT}} \times e^{r \times 3/12} \\
&= \frac{(20)(0.25)}{0.633} \times e^{0.16/4} + \frac{-4.5 e^{-0.12 \times 1/4}}{0.633} \times e^{0.12/4} \\
&= 1.112258
\end{aligned}
$$

Thus, the discount rate for the call is 42.56% per annum.

Pricing Options Using DCF

Q3. Calculate the option price in Q2 using DCF. Is the price the same as the price from the risk-neutral valuation?

Risk-Neutral Valuation vs. DCF

- The option price from the risk-neutral valuation is **the same** as the price from the DCF.
- If required return on stock α is higher than the risk-free rate r, it follows that $p < p^*$.
- It implies that in the risk-neutral valuation, we amplify the probability of a bad outcome for stock investors, i.e, $1 - p$.
- We interpret this as the probability being modified to incorporate investors' risk-aversion.

Risk-Neutral Valuation

- Risk-neutral valuation is an **interpretation** of the option pricing formula obtained from the replicating portfolio.

- This does not mean that invetors are risk neutral!

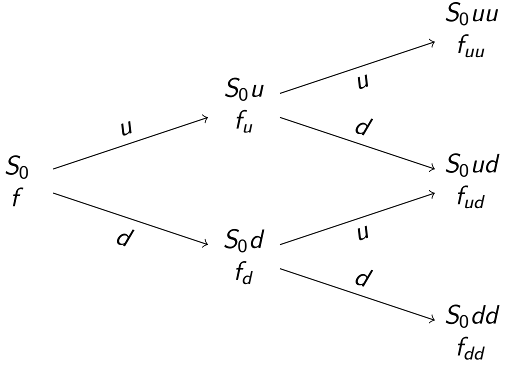
- We incorporate risk aversion in two ways:
    - Add risk premium to the cost of capital.
    - Increase the probability of the bad states.

- This interpretation is also useful for multi-step binomial models.

Two-Step Binomial Models

Two-Step Binomial Model

- The current stock price is S_0 and may go up by u or down by d in a time step. Each time step is Δt and the risk-free interest rate is r per annum.
- A European call option has the strike price of K and expires in two steps. What is the option price?



Two-Step Binomial Model

- We start at the option expiration date and find the option payoff at each stock price then.
- At $T = \Delta t$, each price and the following prices can be seen as one-step binomial tree. Thus, we can use the pricing formula of one-step models.

$$\begin{cases} f_u &= e^{-r\Delta t}\left[pf_{uu} + (1-p)f_{ud}\right] \\ f_d &= e^{-r\Delta t}\left[pf_{ud} + (1-p)f_{dd}\right] \end{cases}$$

where $p = \frac{e^{r\Delta t} - d}{u - d}$.

Two-Step Binomial Model

- At $T = 0$,

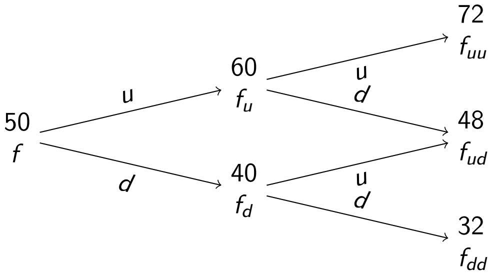
$$\begin{aligned} f_0 &= e^{-r\Delta t}\left[pf_u + (1-p)f_d\right] \\ &= e^{-r\Delta t}\left[p\left(e^{-r\Delta t}\left[pf_{uu} + (1-p)f_{ud}\right]\right) + (1-p)\left(e^{-r\Delta t}\left[pf_{ud} + (1-p)f_{dd}\right]\right)\right] \\ &= e^{-2r\Delta t}\left[p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\right] \end{aligned}$$

- This is consistent with the probabilistic interpretation of p.
  - In risk-neutral valuation, p^2, $2p(1-p)$, and $(1-p)^2$ are probabilities of reaching top, middle, and bottom final nodes.

Two-Step Binomial Model - Put Option

- To price a put option, we use put payoffs at the option expiration. The rest of calculation is the same as the call valuation.

Q. Consider a 2-year European put with K=\$52 on a stock with S_0=\$50. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 20%. The risk-free interest rate is 5%. What is the price of the put option?



N-Step Binomial Models

N-Step Binomial Model

- Suppose that there are N time steps until the option maturity and each time step is Δt.

- The risk-neutral probability of an increase in stock price during each step is $p(=\frac{e^{r\Delta t}-d}{u-d})$.

- There are $N+1$ nodes at the expiration. Let node j denote the final stock price when the price moves upward j times and downward $N-j$ times. There, the final stock price would be

$$S_0 u^j d^{N-j},$$

where $j = 0, 1, ..., N$.

N-Step Binomial Model

- To determine the price of an European option, we need the probability of reaching each node at the expiration.

- The probability of reaching the node j is

$$\binom{N}{j} p^j (1-p)^{N-j}$$

  - There are multiple paths leading to the node j. The number of the paths is $\binom{N}{j}$, which is j-combinations from a set of N elements.
  - How to calculate $\binom{N}{j}$?
    - In algebra, $\binom{N}{j} = \frac{N!}{j!(N-j)!}$.
    - In Excel, use "combin(N,j)".

N-Step Binomial Model

- For each node, the probability and option payoff is as follows:

No. of up	No. of down	Probability	Stock price at T	Option payoff
0	N	$p^0(1-p)^N$	$S_0 u^0 d^N$	f_0
1	$N-1$	$\binom{N}{1} p^1(1-p)^{N-1}$	$S_0 u^1 d^{N-1}$	f_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$N-j$	$\binom{N}{j} p^j(1-p)^{N-j}$	$S_0 u^j d^{N-j}$	f_j
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	0	$p^N(1-p)^0$	$S_0 u^N d^0$	f_N

- The price of European option is then

$$e^{-r(N\Delta t)} \sum_{j=0}^{N} \binom{N}{j} p^j (1-p)^{N-j} f_j$$

where f_j is the option payoff at node j.

N-Step Binomial Model

Q. Consider a 3-year European call with K=\$30 on a stock with S_0=\$30. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 10%. The risk-free interest rate is 5%. What is the price of the call option?

N-Step Binomial Model

Q. Consider a 3-year European call with K=\$30 on a stock with S_0=\$30. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 10%. The risk-free interest rate is 5%. What is the price of the call option?

Answer: First, the option payoffs at each of 4 nodes are

$$
\begin{aligned}
f_0 &= \max(30(1.1)^0(0.9)^3 - 30, 0) = 0 \\
f_1 &= \max(30(1.1)^1(0.9)^2 - 30, 0) = 0 \\
f_2 &= \max(30(1.1)^2(0.9)^1 - 30, 0) = 2.67 \\
f_3 &= \max(30(1.1)^3(0.9)^0 - 30, 0) = 9.93.
\end{aligned}
$$

The risk-neutral probability is $p = \frac{e^{0.05\times 1} - 0.9}{1.1 - 0.9} = 0.756$. Then, the option price is

$$
e^{-0.05\times 3} \sum_{j=0}^{3} \binom{N}{j} (0.756)^j (1 - 0.756)^{N-j} f_j = 4.01
$$

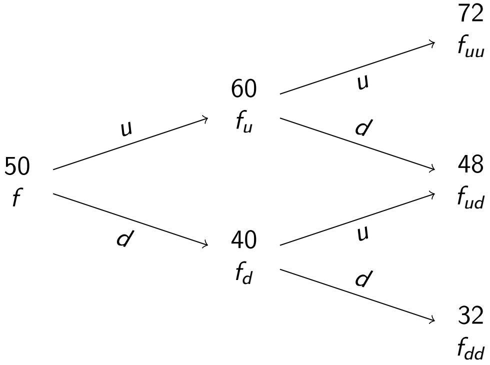
Pricing American Options

American Options

- In pricing American options, we should consider that the options can be exercised early.
- In a similar way to pricing European options, we build a binomial tree of stock price. Then, we start from final nodes and proceed backward.
- However, the option value at each node becomes the maximum of
  1. the option value when delaying the exercise,
  $$f = e^{-r\Delta t}\left[pf_u + (1-p)f_d\right]$$
  2. the payoff when exercising now

American Options

e.g. Consider a 2-year American put with K=\$52 on a stock with S_0=\$50. Suppose that a time step is 1 year, and in each time step the stock price moves either up or down by 20%. The risk-free interest rate is 5%. What is the price of the put option?



American Options

- The risk-neutral probability is

$$p = \frac{e^{0.05\times 1} - 0.8}{1.2 - 0.8} = 0.6282.$$

- At final nodes, option payoffs are

$$\begin{cases} f_{uu} & = \max(52 - 72, 0) & = 0 \\ f_{ud} & = \max(52 - 48, 0) & = 4 \\ f_{dd} & = \max(52 - 32, 0) & = 20 \end{cases}$$

American Options

- At top node in $T = 1$, the option price is

$$f_u = \max\left(\underbrace{e^{-r\Delta t}\left[pf_{uu} + (1-p)f_{ud}\right]}_{\text{value if delay}}, \underbrace{52 - 60}_{\text{value if exercise}} \right) = \$1.415.$$

  At bottom node in $T = 1$, the option price is

$$f_d = \max\left(e^{-r\Delta t}\left[pf_{ud} + (1-p)f_{dd}\right], 52 - 40\right) = \$12.$$

- At the initial node, the option price is

$$f = \max\left(e^{-r\Delta t}\left[pf_u + (1-p)f_d\right], 52 - 50\right) = \$5.090.$$

Determining u and d

Determining u and d

- We have studied how to price options when u and d are given in binomial trees.
- If they are not given, how can we determine u and d?
- For this determination, we focus on the volatility of underlying asset.
  - The volatility σ is the standard deviation of yearly returns on the stock.
- The basic idea is to choose u and d such that the volatility in the binomial tree matches the volatility we see in data.

Determining u and d

- Once the volatility σ is obtained from data, we want to construct a binomial tree such that returns in the tree have the same volatility.
- This means that the return over one day $(= \Delta t)$ should have the standard deviation of $\sigma\sqrt{\Delta t}$
  - When $\Delta t = 1/365$, $\sigma_{\Delta t} = \sigma\sqrt{1/365}$, ($\sigma$ is an annual std.dev.).
- We can achieve this by choosing

$$u = e^{\sigma\sqrt{\Delta t}} \quad \text{and} \quad d = e^{-\sigma\sqrt{\Delta t}}$$

  in the binomial tree.
- Why does this choice of u and d work?

Determining u and d

- Suppose that the required return on the stock in the real world is α per annum. Then, the real probability $p^* = \frac{e^{\alpha \Delta t} - d}{u - d}$.
- Using this real probability, we can compute the variance of return $\text{Var}(r)$.
- We want to show that $\text{Var}(r)$ equals $\sigma^2 \Delta t$ under this particular choice of u and d.

$$\begin{aligned}
\text{Var}(r) &= E(r^2) - [E(r)]^2 \\
&= p^*(u-1)^2 + (1-p^*)(d-1)^2 - [p^*(u-1) + (1-p^*)(d-1)]^2 \\
&= \dots \\
&= (u+d)e^{\alpha \Delta t} - ud - e^{2\alpha \Delta t}
\end{aligned}$$

Determining u and d

- Now, let's plug in $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$.

$$\begin{aligned}\text{Var}(r) &= (e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})e^{\alpha\Delta t} - 1 - e^{2\alpha\Delta t} \\ &= \ldots?\end{aligned}$$

Determining u and d - Math Review

- Taylor series
  - A function $f(x)$ can be expressed as a sum of polynomials:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

  - A Taylor series of e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

  - When x is small, we can ignore higher-order terms.

Determining u and d

- Applying the Taylor series to the exponential terms, we can simplify the variance.
- Here, we assume that Δt is very small, so we ignore $\Delta t^{3/2}$, Δt^2, and higher powers.
  - For instance, $e^{\sigma\sqrt{\Delta t}} \approx 1 + \sigma\sqrt{\Delta t} + \frac{(\sigma\sqrt{\Delta t})^2}{2}$.
- Then, the variance becomes

$$
\begin{aligned}
\mathrm{Var}(r) &= (e^{\sigma\sqrt{\Delta t}} + e^{-\sigma\sqrt{\Delta t}})e^{\alpha\Delta t} - 1 - e^{2\alpha\Delta t} \\
&\approx (2 + \sigma\sqrt{\Delta t} - \sigma\sqrt{\Delta t} + \sigma^2\Delta t)(1 + \alpha\Delta t) - 1 - (1 + 2\alpha\Delta t) \\
&\approx \sigma^2\Delta t
\end{aligned}
$$

Options on Stocks Paying a Continuous Dividend Yield

- Consider a stock paying a known dividend yield at rate q. E.g. stock indices.
- The total return from **dividends** and **capital gains** in a risk-neutral world is r.
- The dividends provide a return of q.
- Capital gains must therefore provide a return of $r - q$.
- If the stock starts at S_0, its expected value after one time step of length Δt must be $S_0 e^{(r-q)\Delta t}$.
- This means, $pS_0 u + (1-p)S_0 d = S_0 e^{(r-q)\Delta t}$.
- Hence, $p = \frac{e^{(r-q)\Delta t} - d}{u-d}$.

NB Options on currencies: $p = \frac{e^{(r-r_f)\Delta t} - d}{u-d}$.

Options on Futures

- Entering a futures contract requires **no initial investment**.
- In a **risk-neutral world**, every asset's expected return equals the risk-free rate *on invested capital*.
  - Since a futures contract requires no capital, its expected return must be zero.
- Therefore, under the risk-neutral measure:

$$E^{\mathbb{P}}[dF_t] = 0 \quad \Rightarrow \quad E^{\mathbb{P}}[F_T] = F_0.$$

- Intuition: a zero-cost, riskless-expected-gain contract would be an arbitrage.
- This means, $pF_0u + (1-p)F_0d = F_0$.
- Hence, $p = \frac{1-d}{u-d}$.

Application of Binomial Model to Corporate Finance

Application to Corporate Finance

- We have studied the binomial model as a tool for pricing options, but this model can be used to understand corporate finance.

- In particular, we can use the binomial model to find the present value of shareholders' equity and liabilities.

- The starting point is the balance sheet identity:

$$\text{Assets} = \text{Liabilities} + \text{Shareholders' Equity}$$

Application to Corporate Finance

- The market value of assets will change over time.
  - This market value is the present value of future cash flows from firms' business. As the business outlook changes time to time, the value also changes.
- Thus, we can consider a binomial tree to model future asset values.
- In this binomial tree, we can determine the present value of liabilities and equity.

Application to Corporate Finance

- Suppose that a firm has liabilities L that are due time T. Let V_T denote the firm's asset value then.

- The payoffs to creditors (debtholders) are

$$\min(V_T, L)$$

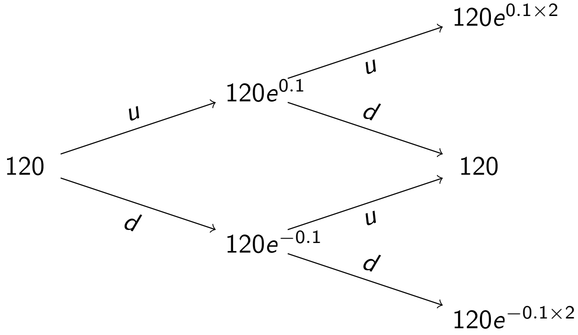
- The payoffs to shareholders (equityholders) are

$$\max(V_T - L, 0)$$

- Once the final payoffs are identified, we can calculate the present values as we do in option pricing.

Application to Corporate Finance - Example

Q. A company's current value of assets is $\$120$ millions and the volatility of the asset value is 10% per annum. The company has issued a bond, so that it needs to repay $\$100$ millions two years later from now. The risk-free interest rate is 5% per annum. What is the current value of shareholders' equity? Use a two step binomial tree.



Application to Corporate Finance - Example

Answer: The payoffs to shareholders are

$$f_{uu} = \max(146.57 - 100, 0) = 46.57$$
$$f_{ud} = \max(120 - 100, 0) = 20$$
$$f_{dd} = \max(98.25 - 100, 0) = 0$$

The risk-neutral probability is $p = \frac{e^{0.05} - e^{-0.1}}{e^{0.1} - e^{-0.1}} = 0.731$. Then, the present value of the equity is

$$e^{-0.05 \times 2} \left[(0.731)^2 (46.57) + 2(0.731)(1 - 0.731)(20) + (1 - 0.731)^2 (0) \right]$$
$$= \$29.63 \text{ millions}$$

Application to Corporate Finance - Example

Q2. Go back to the Q1. What is the present value of debt?