Properties of Options

BUSS386. Futures and Options

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Lecture Outline

- Characterizing Option Prices
    - Factors Affecting Option Prices
    - Lower and Upper Bounds for Option Prices
    - Put-Call Parity
- Reading: Chp. 11

Factors Affecting Option Prices

Question: How to Price Options?

- In this lecture, we characterize option prices as much as possible using the no-arbitrage argument.

- First, we want to know what factors matter for option prices.

Question: How to Price Options?

- From Financial Management, we learned the valuation tool, **Discounted Cash Flow**.

- According to the DCF, the option price is the present value of future option payoff.

- Consider a European call option. Then, the option price is

$$\underbrace{e^{-r_{\text{call}}T}}_{\text{discounting factor}} \times \underbrace{E\left[\max(S_T - K, 0)\right]}_{\text{option payoff}}$$

- Thus, factors affecting the option payoff or discounting will also affect the option price.

Factors Affecting Option Prices

- The following 6 factors matter for the current option price in time 0.

1. Current stock price, S_0
2. Strike price, K
3. Volatility of stock price, σ
4. Time to expiration, T
5. Risk-free interest rate, r
6. Dividends expected to be paid.

Factors Affecting Option Prices - Stock Price and Strike Price

	European call	European put	American call	American put
Current stock price	$+$	$-$	$+$	$-$
Strike price	$-$	$+$	$-$	$+$

- The payoff of the call option increases with the stock price.
- The payoff of the call option decreases with the strike price K.
- The payoff of the put option behaves in the opposite way. Thus, the put price increases with K and decreases with S_0.

Factors Affecting Option Prices - Volatility

	European call	European put	American call	American put
Volatility	+	+	+	+

- Volatility is a measure of uncertainty in future stock price movement (e.g. the standard deviation).
- As volatility increases, the chance that the stock will do very well or very poorly increases.
- The owner of a call benefits from price increases but has limited downside risk in the event of price decrease.
- The owner of a put benefits from price decreases but has limited downside risk in the event of price increase.

Factors Affecting Option Prices - Future Dividends

	European call	European put	American call	American put
Amount of future dividends	$-$	$+$	$-$	$+$

- When the current stock price is fixed, more future dividends until the option expiration lowers the future stock price on the expiration date.
- The call price decreases with the expected amount of future dividends.
- The put price increases with the expected amount of future dividends.

Factors Affecting Option Prices - Time to Expiration

	European call	European put	American call	American put
Time to expiration	?	?	$+$	$+$

- For American options, options with a longer time to expiration are worth more or as much as options with a shorter time to expiration.
- This is because that the owner of a long-life option has all exercise opportunities open to the owner of a short-life option and more.
- For European options, generally, they are more valuable with a longer time to expiration.
  - However, there are some exceptions, for example, when a large dividend is expected to be paid within the time to expiration or when a company is likely to default.

Factors Affecting Option Prices - Risk-Free Rate

	European call	European put	American call	American put
Risk-free rate	$+$	$-$	$+$	$-$

- An increase in interest rate will decrease the PV of strike price.
- For call options, the expected payoff is $E[S_T] - PV(K)$. Hence, the option becomes more valuable with a higher interest rate.
- For put options, the expected payoff is $PV(K) - E[S_T]$. Hence, the option becomes less valuable with a higher interest rate.

Factors Affecting Option Prices

Q. Consider two European put options with the same strike prices and the same expiration dates but for different underlying stocks A and B. Stock A has the volatility of 20% and is expected to pay dividend \$3 in a year. The stock B has the volatility of 15% and is expected to pay dividend \$2 in a year. The current stock prices of A and B are the same. Can we determine which put option has the higher price?

Lower/Upper Bounds for Option Prices

Put-Call Parity

Properties of Stock Options

- To exactly determine the option price, we need a model that describes futures stock price.
- We don't want to use a model yet.
- Instead, we use the no-arbitrage argument only and try to characterize the option price as much as possible.
- This results in ...
  - Lower/upper bound for the option price
  - Put-call parity

Assumptions and Notation

- Assumptions

  1. There are no transaction costs.
  2. All trading profits are subject to the same tax rate.
  3. Borrowing and lending are possible at the risk-free interest rate.

- Notation

  - S_0: Current stock price
  - K: Strike price
  - T: Expiration date
  - S_T: Stock price at expiration
  - r: Risk-free interest rate (continuously compounding)
  - C_0: Value of American call option
  - P_0: Value of American put option
  - c_0: Value of European call option
  - p_0: Value of European put option

European vs American Options

- Consider an American option and a European option with the same strike prices, expiration dates, and underlying assets.

- The American option is always more valuable or as valuable as the European option.

$$c_0 \leq C_0 \quad \text{and} \quad p_0 \leq P_0$$

- This is because the owner of the American option has all exercise opportunities open to the owner of the European option and more.

- This relationships hold for all types of options irrespective of whether the underlying asset pays dividends or not.

Properties of Stock Options

- We consider lower/upper bounds and the put-call parity case by case.

1. Non-dividend-paying stock
    - European options
    - American options
2. Dividend-paying stock
    - Continuous dividend
        - European options
        - American options
    - Discrete dividend
        - European options
        - American options

Lower/Upper Bounds for Option Prices

Put-Call Parity

I. Options on Non-Dividend-Paying Stocks

Upper Bounds for Option Prices - Call

- For call options on non-dividend-paying stock,

$$c_0 \leq S_0 \quad \text{and} \quad C_0 \leq S_0$$

- Why?
  - Let's compare the stock and the European call in terms of time-T cash flow:

    $$\begin{aligned} &\text{Stock}: && S_T \\ &\text{European call}: && \max(S_T - K, 0) \end{aligned}$$

    Because $S_T \geq \max(S_T - K, 0)$, we conclude that $S_0 \geq c_0$.

  - We can obtain the same inequality by comparing time-t cash flows of American call and stock, where $t \, (t \leq T)$ is the time to exercise the call.

Upper Bounds for Option Prices - Put

- For put options on non-dividend-paying stock,

$$p_0 \leq Ke^{-rT} \quad \text{and} \quad P_0 \leq K$$

- Why?
  - Let's compare the bond and the European put in terms of time-T cash flow:

Bond that will pay K at T:	K
European put :	$\max(K - S_T, 0)$

    Because $K \geq \max(K - S_T, 0)$, we conclude that $Ke^{-rT} \geq p_0$.
  - We can obtain $Ke^{-rt} \geq P_0$ by comparing time-t cash flows of American put and a bond that will pay K at t. Because $0 \leq t \leq T$, we conclude that $K \geq P_0$.

Lower Bounds for Option Prices - European Call

- For a European call on a non-dividend-paying stock

$$c_0 \geq S_0 - Ke^{-rT}$$

- Why? Consider the following two portfolios:
  1. European call
  2. one share + sell bond with face value of K at T

- At the option expiration T, ① always generates larger cash flows than ②:
  1. $\max(S_T - K, 0)$
  2. $S_T - K$

- Hence, under no-arbitrage, current prices should satisfy

$$c_0 \geq S_0 - Ke^{-rT}$$

Lower Bounds for Option Prices - European Call

- Combining the fact the option value cannot be negative,

$$c_0 \geq \max(S_0 - Ke^{-rT}, 0)$$

- If the above bound does not hold, an arbitrage exists.
- To make an arbitrage, we sell high and buy low.

Lower Bounds for Option Prices - European Call - Example

Q. Suppose that a call option with $K = \$18$, $r = 10\%$, and $T = 1$ is priced at \$3.00. The current stock price is $S_0 = \$20$. Is there an arbitrage opportunity?

$\Rightarrow$ To prevent any arbitrage, the call price should be between the lower and upper bounds. The upper bound is 20. The lower bound is

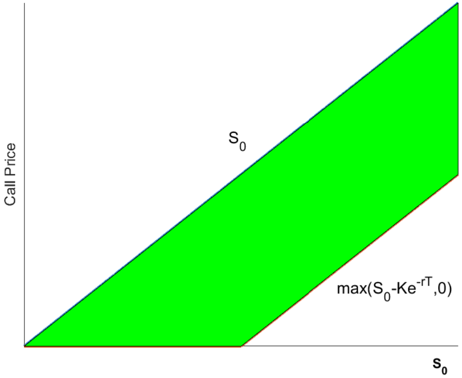
$$\max\left(20 - 18e^{-0.1}, 0\right) = 3.713$$

The option price is lower than the lower bound, so an arbitrage opportunity exists. The arbitrage strategy is

Action today	Cash flow now	Cash flow at T	
		$S_T \geq 18$	$S_T < 18$
long call	-3	$(S_T - 18)$	0
buy a bond	-18$e^{-0.1}$	18	18
sell a share (short-selling)	20	-S_T	-S_T
net	0.713	0	18-S_T

Upper/Lower Bounds for Option Prices - European Call

- Combining the upper and the lower bound, we can find a range of call prices that satisfy the no-arbitrage condition.



Lower Bounds for Option Prices - European Put

- For a European put on a non-dividend-paying stock

$$p_0 \geq Ke^{-rT} - S_0$$

- Why? Consider the following two portfolios:

  3. European put
  4. bond that will pay K at T + short one share

- At the option expiration T, ③ always generates larger cash flows than ④:
  3. $\max(K - S_T, 0)$
  4. $K - S_T$

- Hence, under no-arbitrage, current prices should satisfy

$$p_0 \geq Ke^{-rT} - S_0$$

Lower Bounds for Option Prices - European Put

- Combining the fact that the option value cannot be negative

$$p_0 \geq \max(Ke^{-rT} - S_0, 0)$$

Q. Consider a European put option with $K = \$40$, $T = 0.5$ years, when $S_0 = \$37$ and $r = 5\%$. The option price is \$1.00. Is there an arbitrage opportunity?

Put-Call Parity

[European Call and Put]

- European put and call options with the same strike price and the same expiration have a special relationship, **put-call parity**.

$$c_0 + Ke^{-rT} = p_0 + S_0$$

- This implies that the value of a European call (put) option can be deduced from the value of a European put (call).

- This relation always holds when there is no arbitrage. Note that this does not depend on any option pricing model (e.g. binomial model or Black-Scholes-Merton model).

Put-Call Parity - Derivation 1

[European Call and Put]

- Consider the previous portfolios:
  1. European call + bond that will pay K at T
  3. European put + one share

- At the option expiration T, ① and ③ always generate the same cash flows:
  1. $\max(S_T - K, 0) + K = \max(S_T, K)$
  3. $\max(K - S_T, 0) + S_T = \max(K, S_T)$

- Hence, under no-arbitrage, current prices should satisfy

$$c_0 + Ke^{-rT} = p_0 + S_0$$

Put-Call Parity - Derivation 2

[European Call and Put]

- Consider a portfolio of options: + long Call - short Put
- The payoff of this portfolio at T is $S_T - K$, which is the same as the payoff of a forward with $F_0 = K$.
- The present value of the payoff is $S_0 - Ke^{rT}$.
- Therefore, $c_0 - p_0 = S_0 - Ke^{-rT}$.

Put-Call Parity - Example

[European Call and Put]

Q. Suppose that $S_0 = \$31$, $K = \$30$, $r = 10\%$, $T = 3$ month. The price of a European call is \$3 and the price of a European put is \$2.25. Is there an arbitrage opportunity? If so, find an arbitrage strategy and its profit.

Answer: Let's check whether the put-call parity holds:

1. $c_0 + Ke^{-rT} = 3 + 30e^{-0.1\times 3/12}$=32.26
3. $p_0 + S_0 = 2.25 + 31 = 33.25$

So, the portfolio ③ is overpriced relative to portfolio ①. Then, the arbitrage strategy is

Action today	Today	T	
		$S_T \geq 30$	$S_T < 30$
long call	-3	$(S_T - 30)$	0
buy a bond	-30$e^{-0.1\times 3/12}$	30	30
short put	2.25	0	$-(30 - S_T)$
sell share	31	$-S_T$	$-S_T$
net	0.99	0	0

American Options - Early Exercise

[Non-dividend-paying stock]

- Long position in an American option has the right to exercise earlier than the expiration.
- In a special case, when we long an **American call on a non-dividend-paying stock**, it is **never optimal to exercise early** before the expiration (for reason we will see below).
- Why is early exercise not optimal for this special case?
  - At time t, the option holder has the value

$$C_t = \max \begin{cases} \text{Value of waiting} & \text{if not exercise} \\ S_t - K & \text{if exercise} \end{cases}$$

American Options - Early Exercise

[Non-dividend-paying stock]

- If $C_t > S_t - K$, we can conclude that early exercise is not optimal.
- It turns out that the above inequality holds at any $0 \leq t < T$.
    - As an American call option is worth more or as much as a European call,

$$C_t \geq c_t$$

    - From the put-call parity of European options, we have

$$\begin{aligned} c_t &= p_t + S_t - Ke^{-r(T-t)} \\ &= p_t + (S_t - K) + K(1 - e^{-r(T-t)}). \end{aligned}$$

      Thus, $c_t > S_t - K$.

    - Hence, $C_t > S_t - K$. It's better to sell the call.

American Options - Early Exercise

[Non-dividend-paying stock]

- Alternatively ...
- Consider the following two portfolios:
  1. American call
  2. One share + short bond that will pay K at T
- At the option expiration T, ① always generates larger cash flows than ②:
  1. $\max(S_T - K, 0)$
  2. $S_T - K$
- Hence, under no-arbitrage, at $t < T$,

$$C_t \geq S_t - Ke^{-r(T-t)} \geq S_t - K$$

American Options - Early Exercise

[Non-dividend-paying stock]

- In the case of an American **put** option, sometimes it is optimal to exercise early.
  - $P_t \leq K$ because the maximum payoff from the option is $K - 0$.
  - Suppose $S_t = 0$, the payoff at t is K and we know $P_t \leq K$.
  - It's better to receive K earlier than later.
  - Hence, it can be desirable to exercise the option at $t \leq T$.

Lower Bounds for American Call

[Non-dividend-paying stock]

- We just proved that the option expiration is the only date that we may exercise an American call on non-dividend-paying stock.
- This means that the European and the American calls will deliver the same cash flows. Thus, $C_0 = c_0$.
- Hence, the lower bound of American call is the same as the lower bound of European call.

Lower Bounds for American Put

[Non-dividend-paying stock]

- For American put option, it is sometimes optimal to exercise early, in particular, when the option is deep in the money.

- At time t, the option holder has the value

$$P_t = \max \begin{cases} \text{Value of waiting} & \text{if not exercise} \\ K - S_t & \text{if exercise} \end{cases},$$

  Thus, we know $P_t \geq K - S_t$.

- Combining the fact that the put option price cannot be negative, the lower bound becomes

$$P_0 \geq \max(K - S_0, 0).$$

Put-Call Parity for American Options

- For American options on non-dividend-paying stocks, the put-call parity is

$$S_0 - K \leq C_0 - P_0 \leq S_0 - Ke^{-rT}$$

- Let's prove the right inequality first and then prove the left inequality.

Put-Call Parity for American Options - Right Inequality

- As $P_0 \geq p_0$, it follows that $C_0 - P_0 \leq C_0 - p_0$. Also, we know that $C_0 = c_0$ for non-dividend-paying stock. Thus,

$$C_0 - P_0 \leq C_0 - p_0 = c_0 - p_0$$

From the put-call parity for European options, we know that $c_0 - p_0 = S_0 - Ke^{-rT}$. Thus,

$$C_0 - P_0 \leq c_0 - p_0 = S_0 - Ke^{-rT},$$

which proves the right inequality.

Put-Call Parity for American Options - Left Inequality

- To prove the left inequality, we consider the following two portfolios:

portfolio A:	American call + bond worth K now
portfolio B:	American put + stock

- We want to prove that the value of the portfolio A is higher than or equal to the value of portfolio B. This will lead to the left inequality, $C_0 - P_0 \geq S_0 - K$.

- In derivation, we consider the two different cases:

  1. case 1: put option is exercised earlier than the expiration.

  2. case 2: put option is not early-exercised.

Put-Call Parity for American Options - Left Inequality - Case 1

- Suppose that the put option is exercised earlier than the expiration, say t $(0 \leq t < T)$.
- Then, the portfolio value at time t is

$$\begin{array}{ll} \text{portfolio A:} & C_t + Ke^{rt} \\ \text{portfolio B:} & (K - S_t) + S_t \end{array}$$

- As $C_t \geq 0$ and $e^{rt} \geq 1$, we can conclude that the time-t value of portfolio A is higher than or equal to the value of portfolio B.

Put-Call Parity for American Options - Left Inequality - Case 2

- Suppose that the put option is NOT exercised earlier than the expiration. Then, the put option may or may not be exercised on the expiration T.

- If $S_T \geq K$ on the expiration T,

$$\begin{array}{ll} \text{portfolio A:} & (S_T - K) + Ke^{rT} \\ \text{portfolio B:} & 0 + S_T \end{array}$$

  Thus, the portfolio A has the higher value.

- If $S_T < K$ on the expiration T,

$$\begin{array}{ll} \text{portfolio A:} & 0 + Ke^{rT} \\ \text{portfolio B:} & K - S_T + S_T \end{array}$$

  Again, the portfolio A has the higher value.

Put-Call Parity for American Options - Left Inequality - Case 2

- Thus, we can conclude that the time-T value of portfolio A is higher than the value of portfolio B.

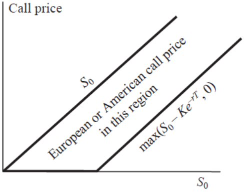
- $C_0 + K \geq P_0 + S_0 \Longrightarrow C_0 - P_0 \geq S_0 - K.$

Summary of Bounds for Option Prices

[Non-dividend-paying stock]

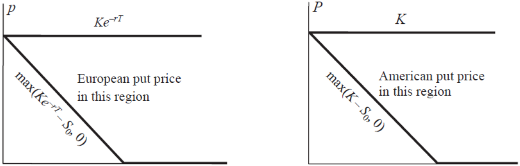
Call on non-dividend-paying stock

Figure 10.3 Bounds for European and American call options when there are no dividends.



Put on non-dividend-paying stock

Figure 10.5 Bounds for European and American put options when there are no dividends.



Lower/Upper Bounds for Option Prices

Put-Call Parity

II. Options on Dividend-Paying Stocks

Properties of Options on Dividend-Paying Stock

- Recall that the underlying asset's dividend payment affects option prices.
- Hence, the bounds and put-call parity should be modified for options on dividend-paying stock.
- For European options on dividend-paying stocks, we can find the result via a shortcut:
  - Starting from the results for non-dividend-paying stocks, we replace S_0 with the ex-dividend component.

Lower/Upper Bounds and Put-Call Parity for European Options - Discrete Dividends

- Suppose that the underlying assets pay discrete dividends. Let D denote the present value of futures dividends until the option expiration.

- For European call,

$$\max\left(S_0 - D - Ke^{-rT}, 0\right) \leq c_0 \leq S_0 - D$$

- For European put,

$$\max\left(Ke^{-rT} - (S_0 - D), 0\right) \leq p_0 \leq Ke^{-rT}$$

- Put-Call parity,

$$c_0 + Ke^{-rT} = p_0 + S_0 - D$$

Lower/Upper Bounds and Put-Call Parity for European Options - Discrete Dividends

Q. A European call option on a stock with $K = 20$ and $T = 3$ is priced at \$9. The current stock price is \$30, and the stock is expected to pay dividend of \$2 in $T = 1$ and $T = 2$. The risk-free interest rate is 3%. What is the price of a European put option with the same strike price and expiration date?

Lower/Upper Bounds and Put-Call Parity for European Options - Continuous Dividends

- Suppose that the underlying assets pay continuous dividends with the dividend yield q per annum.

- For European call,

$$\max\left(S_0 e^{-qT} - Ke^{-rT}, 0\right) \leq c_0 \leq S_0 e^{-qT}$$

- For European put,

$$\max\left(Ke^{-rT} - S_0 e^{-qT}, 0\right) \leq p_0 \leq Ke^{-rT}$$

- Put-Call parity,

$$c_0 + Ke^{-rT} = p_0 + S_0 e^{-qT}$$