Pricing Forwards and Futures

BUSS386. Futures and Options

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Lecture Outline

- No Arbitrage Argument
  - What is arbitrage?
  - Short-selling
- Determination of Forward Prices
- Valuing Forward Contracts
- Comparison Between Forward and Futures Prices
- Reading: Ch. 5

No Arbitrage Argument

Arbitrage

- **Arbitrage** is a trade where investors can make "free lunch" profits.
- For instance, if we see a price difference for the same assets, we can make an arbitrage profit (buy low and sell high).

e.g. Suppose that a stock is traded in both New York Stock Exchange and London Stock Exchange. Its price in New York is \$140, while it is £100 in London. The exchange rate is \$1.43 per pound.

  - Buy a share in New York and sell it in London.
  - Profit = 100×1.43 - 140 = \$3. This profit is risk-free.

Arbitrage - Definition

- Formally, we claim that a trading strategy is an arbitrage if it satisfies the following conditions.
  1. It always generates **non-negative** cash flows, and
  2. It sometimes generates **positive** cash flows.

- Is each of the following strategies an arbitrage?

	Cash flows $T = 0$	$T = 1$	$T = 2$
Strategy 1	-2	1	3
Strategy 2	0	0	0.5

	Cash flows $T = 0$	$T = 1$	
		case 1	case 2
Strategy 3	0	0.2	0.2
Strategy 4	0	0	0.3

No Arbitrage Argument

- In the markets, there are numerous investors looking for any arbitrage opportunity.
- Suppose that an arbitrage exists for a certain asset.
- Due to forces of supply and demand, the prices will eventually change. In equilibrium, the prices of one asset will be the same across different markets.
- Generally, arbitrage opportunities quickly disappear.

No Arbitrage Argument

- Also, we can apply the no arbitrage argument to two assets (portfolios) A and B that will generate the same cash flows in the future in every condition.

- The current prices of assets A and B should be the same. Otherwise, an arbitrage exists.

- If current prices are different, we can make an arbitrage through "**buy low and sell high**".

  $\Rightarrow$ However, an arbitrage should NOT exist in a purely competitive financial market.

Arbitrage - Assumptions

In making an arbitrage strategy, we assume the followings.

- We consider an investor who has nothing in hand at the beginning of the strategy and liquidates all assets at the end.
- We measure profit/loss in terms of **cash flows**.
- The investor can borrow money (sell a bond) or lend money (buy a bond) at the risk-free rate.
  - We can choose any bond amount (face value) as we like.

e.g. If we buy a bond at the rate r,

Action	time 0	time T
buy a bond (lend money)	-1	e^{rT}

Short Selling

- In constructing an arbitrage, we assume that the market allows short selling.

Def. Short selling is selling an asset that we do not own.

e.g. Suppose that an investor wants to short a stock at time 0 at the current price of \$120.

  - At time 0, the investor borrows the stock, sells immediately, and receives the proceeds of \$120.
  - One year later, stock price falls to \$100. To close the position, the investor buys the stock and return it back to the original owner.

Action	year 0	year 1
(Short) Sell a stock	120	-100

Short Selling

- What if the share pays dividend?
- Then, the shorting investor needs to pay the dividend to the original owner.

e.g. An investor shorts a stock at time 0 whose current price is \$120. The stock pays \$5 dividend in six month.

- Again, by borrowing and selling immediately, the investor receives \$120.
- In six month, the investor provides the original owner with the \$5 dividend.
- One year later, stock price falls to \$90. To close the position, the investor buys the stock and return it back to the original owner.

Action	year 0	year 0.5	year 1
(Short) Sell a share	120	-5	-90

Determination of Forward Prices

Determination of Forward Prices - Basic Idea

- Investors enter a long or short position in forward contract **at zero cost**.
- In other words, the value of forward contract should be **zero** at the time of initiating the contract.
- Conversely, we can determine the forward price, so that the current value of forward contract becomes zero.

Determination of Forward Prices - Setting

- Assumptions
  - No transaction costs.
  - The market participants have the same tax rate on all net trading profits.
  - The market participants can borrow or lend money at the risk-free interest rate.
  - The market participants take advantage of arbitrage opportunities.

- Notation
  - T: delivery date of contract
  - S_0: spot price of the underlying asset today
  - S_T: spot price of the underlying asset at time T
  - F_0: forward price today
  - r: risk-free rate per annum (with continuous compounding)

Determination of Forward Prices

- Consider an underlying asset that pays no dividends. Its current price is S_0.
- What should be the forward price?

Determination of Forward Prices - Derivation 1

- Your goal is to own a stock at T.
  1. long forward with F_0.
  2. borrow S_0, buy a stock, and wait til T.

- At the contract maturity T, the two strategies should have to same cash flow.
  1. $S_T - F_0$
  2. $S_T - S_0 e^{rT}$

- No net cash flow today. Therefore:

$$F_0 = S_0 e^{rT}$$

Determination of Forward Prices - Derivation 2

- Let's consider the following two portfolios:
  1. long forward with F_0 + buy a bond that will pay F_0 at T
  2. buy a stock

- At the contract maturity T, the two portfolios have the same cash flows:
  1. $(S_T - F_0) + F_0$
  2. S_T

- Thus, their current value should be the same:

$$0 + F_0 e^{-rT} = S_0$$

Determination of Forward Prices - Derivation 3

- The payoff from a forward contract at T is $S_T - F_0$
- The present value of the payoff at time 0 is $S_0 - F_0 e^{-rT}$
  - $S_0 =$ PV of $S_T = e^{\alpha T} S_T$, where α is the discount rate accounting for the risk of the stock.
- The value of the forward is zero at 0. Therefore, $0 = S_0 - F_0 e^{-rT}$
- Solving for $F_0 = S_0 e^{rT}$

Determination of Forward Prices - Arbitrage

- What if

$$F_0 \neq S_0 e^{rT}?$$

$\Rightarrow$ An arbitrage exists.

e.g. Consider a 3-month forward contract on a stock whose current price is \$40. The 3-month risk-free interest rate is 5% per annum.

1. What if the forward price is 43 ($> 40e^{0.05\times 3/12}$)?

   $\Rightarrow$ There is an arbitrage:

Action	Cash flow in 0	Cash flow in 3 month
buy stock	-40	S_T
short forward	0	$43 - S_T$
sell bond	40	$-40e^{0.05\times 3/12}$
net	0	2.497

Determination of Forward Prices - Arbitrage

2. What if the forward price is 39 ($< 40e^{0.05\times3/12}$)?

   $\Rightarrow$ There is another arbitrage strategy:

Action	Cash flow in 0	Cash flow in 3 month
sell stock (short selling)	40	$-S_T$
buy forward	0	$S_T - 39$
buy bond	-40	$40e^{0.05\times3/12}$
net	0	1.503

Determination of Forward Prices - Example

Q. Consider a 1-year forward contract on a stock whose current price is \$50. The forward price is \$51, and the risk-free interest rate is 7% per annum. Is there an arbitrage? If so, show the arbitrage strategy.

Action	Cash flow in 0	Cash flow in 1 year
net		

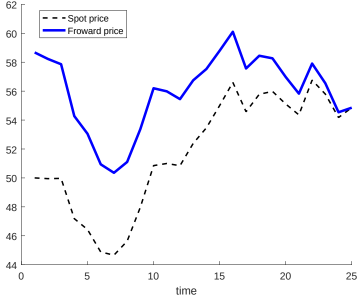
Forward and Spot Prices

- Consider a forward contract initiating at time t. Given the maturity date T, the forward price is

$$F_t = S_t e^{r(T-t)}$$

- Thus, the forward and spot prices are usually different. Only at the expiration, they become the same.

- Also, the forward price changes through time.



Determination of Forward Prices for Underlying Assets Paying Dividends

Dividend Payment and Forward Prices

- Until now, we have assumed that the underlying assets in forward do not pay any dividends.

- What if the underlying asset will pay dividends in the future? Are there changes in forward prices?

⇒ Yes, because...

  - The current price S_0 of the underlying asset includes the future dividends.

  - However, a long/short position in forward will not receive the dividends. Also, the forward payoff is determined by the ex-dividend price.

Determination of Forward Prices - Discrete Dividends

- We consider two different forms of dividend payments.
  1. Discrete dividends: dividends will be paid at certain points in time.
  2. Continuous dividends: dividends will be paid at every instant continuously.

- We first consider the case of discrete dividends.

- Suppose that stock pays dividends until the maturity T. The present value of all future dividends is I.

- The forward price is

$$F_0 = (S_0 - I)e^{rT}$$

Determination of Forward Prices - Discrete Dividends

- Why? Consider the following two portfolios:
  1. long forward with F_0 + buy a bond that will pay $F_0 + Ie^{rT}$ at T
  2. buy a stock
- At the contract maturity T, the two portfolios have the same cash flows:
  1. $(S_T - F_0) + F_0 + Ie^{rT}$
  2. $(S_T + Ie^{rT})$
- The portfolio values are the same at T. Thus, their current values are the same:

$$0 + F_0 e^{-rT} + I = S_0$$

Determination of Forward Prices - Discrete Dividends

Q1. Consider a 9-month forward contract on a corporate bond. The current price of the corporate bond is \$900, and it will pay \$40 coupon in 4 months. The 4-month and 9-month risk-free rates are 3% and 4%, respectively. If there is no arbitrage, what is the forward price?

Answer: The forward price is

$$(900 - 40e^{-0.03\times 4/12})e^{0.04\times 9/12} = 886.60$$

Determination of Forward Prices - Discrete Dividends

Q2. Consider the 9-month forward contract on the corporate bond in Q1. Suppose that the forward price is \$910. Is there an arbitrage? If so, show the arbitrage strategy.

Determination of Forward Prices - Discrete Dividends

Q2. Consider the 9-month forward contract on the corporate bond in Q1. Suppose that the forward price is \$910. Is there an arbitrage? If so, show the arbitrage strategy.

Answer: $886.60 < 910$. Thus, we can think of the following arbitrage strategy:

Action	Cash flow in 0	Cash flow in 4 month	Cash flow in 9 month
buy corporate bond	-900	40	S_T
short forward	0	0	$910 - S_T$
sell 4-month bond	$40e^{-0.03\times 4/12}$	-40	0
sell 9-month bond	$910e^{-0.04\times 9/12}$	0	-910
net	22.707	0	0

Determination of Forward Price - Continuous Dividends

- Some securities pay continuous dividends (e.g, stock index, foreign currency).
  - Once we invest in a stock index, dividends from each individual stock will be paid at different points of time.
  - Having a lot of stocks in the index, we can approximate the index as paying dividends continuously.
- To simplify the argument, we assume that the dividends will be reinvested immediately to buy more shares.

Determination of Forward Price - Continuous Dividends

- Let q denote the dividend yield per annum. Stock price at time 0 is S_0.
  - Let N denote the number of dividend payments in a year.
  - In one period, investor receives dividend $\frac{q}{N} S_t$.
  - Reinvesting the dividend, the investor owns $\frac{q}{N}$ additional shares. Thus, the number of shares increases by factor of $\left(1 + \frac{q}{N}\right)$ in one period.
  - When investing for one year, the number of shares increases by factor of $\left(1 + \frac{q}{N}\right)^N$. If N becomes infinitely large, the factor becomes e^q.
- If we invest for T years, the number of shares increases by e^{qT}.

Determination of Forward Price - Continuous Dividends

- What if the underlying asset pays continuous dividends with dividend yield q per annum?

- Forward price is

$$F_0 = S_0 e^{(r-q)T}$$

- Why? Consider the two portfolios:
  1. long forward with F_0 + buy a bond that will pay F_0 at T
  2. buy e^{-qT} share of stock

- The two portfolios will have the same cash flows at T:
  1. $(S_T - F_0) + F_0$
  2. $S_T e^{-qT} e^{qT}$

- Therefore, the two portfolios should have the same present values:

$$0 + F_0 e^{-rT} = S_0 e^{-qT}$$

Determination of Forward Price - Continuous Dividends - Foreign Currency

- If we hold a foreign currency, we receive interests that are paid continuously at the risk-free rate prevailing in the foreign country.

- Thus, foreign currency can be regarded as an asset with continuous dividends.

- Forward price is then

$$F_0 = S_0 e^{(r - r_f)T}$$

  where r_f is the foreign risk-free rate.

Determination of Forward Price - Continuous Dividends - Foreign Currency

Q1. Suppose that the 2-year interest rates in Hong Kong and the United Kingdom are 3% and 1%, respectively, and the spot exchange rate between the British Pound (GBP) and the Hong Kong Dollar (HKD) is 9.65 HKD per GBP. A 2-year forward exchange rate is 11.00. Is there an arbitrage for Hong Kong investors?

Determination of Forward Price - Continuous Dividends - Foreign Currency

Q1. Suppose that the 2-year interest rates in Hong Kong and the United Kingdom are 3% and 1%, respectively, and the spot exchange rate between the British Pound (GBP) and the Hong Kong Dollar (HKD) is 9.65 HKD per GBP. A 2-year forward exchange rate is 11.00. Is there an arbitrage for Hong Kong investors?

Answer: $11.00 > 9.65e^{(0.03-0.01)\times 2}$. Thus, there is an arbitrage. We can consider the following strategy:

Action	Cash flow now	Cash flow in 2 year
buy $e^{-0.01\times 2}$ GBP	$-9.65e^{-0.01\times 2}$	$e^{-0.01\times 2}e^{0.01\times 2}S_T$
short forward	0	$11.00 - S_T$
sell HK bond	$11.00e^{-0.03\times 2}$	-11.00
net	0.900	0

Determination of Forward Price - Continuous Dividends - Foreign Currency

Q2. Suppose that the 2-year interest rates in Hong Kong and the United Kingdom are 3% and 1%, respectively, and the spot exchange rate between the British Pound (GBP) and the Hong Kong Dollar (HKD) is 9.65 HKD per GBP. A 2-year forward exchange rate is 9.70. Is there an arbitrage for Hong Kong investors?

Determination of Forward Price - Commodities

- Storing commodities has costs and benefits T
- Forward price with proportional storage cost u

$$F_0 = S_0 e^{(r+u)T}$$

- Forward price with convenience yield y

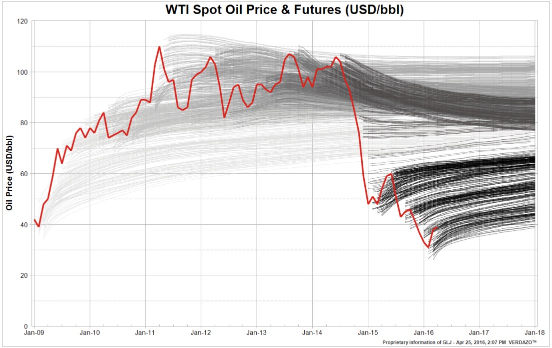
$$F_0 = S_0 e^{(r-y)T}$$

- Together

$$F_0 = S_0 e^{(r-y+u)T}$$

The shape of the forward curve

- Contango is a pattern of forward prices that increases with contract maturity
- Backwardation is a pattern of forward prices over time that decreases with contract maturity



Commodities that cannot be stored

- May be no storage or very limited storage life: electricity, lettuce, strawberries, temperature, rainfall, ...
- For non-storable commodities, forward prices can have information about future spot prices because no-arbitrage conditions don't hold (i.e. not tied to current price)
    - Approach to pricing is to model stochastic future spot prices
    - Also must infer discount rates

Summary

- For stocks, bonds, currencies, metals, stored agricultural commodities, etc., there is no new information in forward prices over what can be learned from spot prices!
- The forward price is tied down by no-arbitrage conditions that depend only on the underlying spot price, interest rates, and associated cash flows between 0 and T (dividends, coupons, storage costs, convenience yield)
- Can we use forward prices to predict future spot price?
- Is the expected future price of a non-dividend paying stock higher or lower than its forward price?

Valuing Forward Contracts

Valuing Forward Contracts

- The value of forward is **zero at the moment we initiate** the contract.
- However, as time passes, its later value can be either negative or positive.
- Suppose that we have a **long position** in a forward with price F_0 that was entered at time 0.
- What is the value f of the forward at time t?

	Value at 0	Value at t
Forward with F_0	0	?

Valuing Forward Contracts

- To find the time-t value of the forward with F_0, we consider another forward that we just start at t.

- Consider the following two portfolios at t:

  1. long forward with F_0 + buy a bond that will pay $F_0 - F_t$ at T
  2. long forward with F_t

- The two portfolios will generate the same cash flows at T:

  1. $(S_T - F_0) + (F_0 - F_t)$
  2. $S_T - F_t$

- Then, the time-t values of the two portfolios should be the same. As a result, the time-t value of the **long position** in forward with F_0 is

$$f + (F_0 - F_t)e^{-r(T-t)} = 0$$

Valuing Forward Contracts

- In a similar way, we can find time-t value of **short position** in forward with F_0 that we started at time 0.

- Consider the two portfolios at t:

  1. short forward with F_0 + buy a bond that will pay $F_t - F_0$ at T
  2. short forward with F_t

- The two portfolios will generate the same cash flows at T:

  1. $(F_0 - S_T) + (F_t - F_0)$
  2. $F_t - S_T$

- Then, the time-t values of the two portfolios should be the same. As a result, the time-t value of the **short position** in forward with F_0 is

$$f = (F_0 - F_t)e^{-r(T-t)}.$$

Valuing Forward Contracts

- We can express the value of forward in a different way by using the forward price F_t

$$F_t = \begin{cases} S_t e^{r(T-t)} & \text{no dividned} \\ (S_t - I)e^{r(T-t)} & \text{discrete divideds} \\ S_t e^{(r-q)(T-t)} & \text{continuous dividends} \end{cases}$$

- As an example, if the underlying asset pays no dividend, the time-t value of the forward is

$$f = S_t - F_0 e^{-r(T-t)}$$

for a long position.

Valuing Forward Contracts

Q. In August 2020, an investor entered a long position in forward on a stock for delivery in August 2021. At that time, stock price was \$40. Two months later, in October 2020, the stock price becomes \$45. What is the value of the forward? Assume that the risk-free rate of interest is 5%.

Forward vs. Futures Prices

Forward vs. Futures Prices

- For the same underlying asset and expiration, the futures and forward prices are very close to each other, but a bit different (due to daily settlement of futures).

- Compare cash flows between forward and futures for a long position:

Day	Forward	Futures
0		
1	0	$F_1 - F_0$
2	0	$F_2 - F_1$
$\vdots$	$\vdots$	$\vdots$
T	$S_T - F_0$	$S_T - F_{T-1}$

- When the risk-free rate is zero, the cumulative gain in futures is the same as the forward payoff. Thus, the forward and futures are the same in cash flows.[1]

  $\Rightarrow$ Futures price = Forward price

[1] In fact, when the interest rate is constant or known, futures price=forward price (Cox, Ingersoll, and Ross, 1981).

Forward vs. Futures Prices

- When the risk-free rate is not zero, the cumulative gain in futures is different from the forward payoff.

Day	Forward	Futures	Interest Factor
0			
1	0	$F_1 - F_0$	$e^{r_1 \times (T-1)/365}$
2	0	$F_2 - F_1$	$e^{r_2 \times (T-2)/365}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	0	$F_t - F_{t-1}$	$e^{r_t \times (T-t)/365}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
T	$S_T - F_0$	$S_T - F_{T-1}$	$e^{r_T \times (T-T)/365}$

- Whether the cumulative gain in futures is larger/smaller than the forward payoff depends on the **correlation** between risk-free rate and spot price of underlying asset.

Forward vs. Futures Prices

1. What if the price of the underlying asset is **positively** correlated with the interest rate?

- For a long position, the gain on futures tend to be **larger** than the forward payoff. Why?
  - Suppose that $S_t > S_{t-1}$. Long position is likely to see daily gain ($F_t - F_{t-1} > 0$). This coincides with a larger interest factor due to a higher interest rate.
  - Suppose that $S_t < S_{t-1}$. Long position is likely to see daily loss ($F_t - F_{t-1} < 0$). This coincides with a smaller interest factor due to a lower interest rate.
- Thus, Futures price > Forward price

Forward vs. Futures Prices

2. What if the price of the underlying asset is **negatively** correlated with the interest rate?

- For a long position, the gain on futures tend to be **smaller** than the forward payoff. Why?
  - Suppose that $S_t > S_{t-1}$. Long position is likely to see daily gain ($F_t - F_{t-1} > 0$). This coincides with a smaller interest factor due to a lower interest rate.
  - Suppose that $S_t < S_{t-1}$. Long position is likely to see daily loss ($F_t - F_{t-1} < 0$). This coincides with a larger interest factor due to a higher interest rate.
- Thus, Futures price $<$ Forward price

Forward vs. Futures Prices

- For most contracts, the covariance between futures prices and interest rates is so low that the difference between futures and forward prices will be negligible.
- However, in contracts on long-term fixed-income securities, prices have a high correlation with interest rates, the covariance can be large enough to generate a meaningful spread between forward and futures prices